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Noncooperative Game Theory

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1 Introduction

Dynamic noncooperative game theory is a field of mathematics and economics in which a lot of research is being carried out at present featuring a great number of applications in many different areas of economics and management science like:

- capital accumulation and investments,
- R&D and technological innovations,
- macroeconomics,
- microeconomics,
- pricing and advertising decisions in marketing,
- natural resource extraction
- pollution control ¹

The aim of this diploma thesis is

- to correct some results for discrete-time affine-quadratic dynamic games of prespecified fixed duration with open-loop and feedback information patterns (cf. Subsections (3.1.3), (3.2.4) and (4.2.3)).
- to give shorter and more convenient proofs for some results already stated in the literature (cf. Subsection (4.1.2)).

¹ For studies in the above-mentioned areas cf. e.g. Dockner et al. (2000) [8].

- to present some extensions for the open-loop and feedback Stackelberg equilibrium solutions of discrete-time affine-quadratic dynamic games of prespecified fixed duration, concerning the number of followers, the structure of the cost und state functions and the possibility of an algorithmic disintegration (cf. Subsections (3.2.2), (4.2.2) and (4.2.5)).

2 Basic Definitions and Basic Insights

2.1 Basic Definitions

In this section the central notions are defined for discrete-time dynamic noncooperative games that will be used permanently throughout the next chapters.²

2.1.1 Game structure

In this subsection definitions are given for the kinds of games, information structures and cost functionals examined in this paper.

Definition 1 *An n -person discrete-time deterministic infinite dynamic game (also known as an n -person deterministic multi-stage game) of prespecified fixed duration involves*

1. *An index set $N := \{1, \dots, n\}$ called the players' set.*
2. *An index set $K := \{1, \dots, T\}$ denoting the stages of the game, where T is the maximum possible number of moves a player is allowed to make in the game*
3. *An infinite set X with some topological structure, called the state set (space) of the game, to which the state of the game (x_{k-1}) belongs for all $k \in K$.*
4. *An infinite set U_k^i with some topological structure, defined for each $k \in K$ and $i \in N$, which is called the action (control) set of player i ($\mathbf{P}i$) at stage k . Its elements are the permissible actions u_k^i of $\mathbf{P}i$ at stage k .*

² The definitions are geared to the ones given in Başar and Olsder (1999)[2]. They were modified insofar as it was helpful to keep the diploma thesis consistent.

5. A function $f_k : X \times U_k^1 \times \dots \times U_k^n \rightarrow X$, defined by

$$x_k = f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n), \quad x_0 \in X(\text{initial state}), \quad k \in K$$

which is called the state equation of the dynamic game. It describes the evolution of the underlying decision process.

6. A set Y_k^i with some topological structure (defined for: $k \in K, i \in N$) called the observation set of \mathbf{Pi} at stage k , to which the observation y_k^i of \mathbf{Pi} belongs at stage k .

7. A function $h_k^i : X \rightarrow Y_k^i$ (defined for: $k \in K, i \in N$) given by

$$y_k^i = h_k^i(x_{k-1}), \quad i \in N, \quad k \in K$$

which is the state-measurement (-observation) equation of \mathbf{Pi} concerning the value of x_{k-1} .

8. A finite set η_k^i (defined for $k \in K, i \in N$) as a subset of $\{y_1^1, \dots, y_k^1; \dots; y_1^n, \dots, y_k^n; u_1^1, \dots, u_{k-1}^1; \dots; u_1^n, \dots, u_{k-1}^n\}$, which determines the information gained and recalled by \mathbf{Pi} at stage k of the game. The specification of η_k^i characterizes the information structure (pattern) of \mathbf{Pi} , and the collection of these information structures for all $i \in N$ is the information structure of the game.

9. A set N_k^i (defined for $k \in K, i \in N$) as a subset of $\{(Y_1^1 \times \dots \times Y_k^1) \times \dots \times (Y_1^n \times \dots \times Y_k^n) \times (U_1^1 \times \dots \times U_{k-1}^1) \times \dots \times (U_1^n \times \dots \times U_{k-1}^n)\}$ designed to be compatible with η_k^i . N_k^i is called the information space of \mathbf{Pi} at stage k , induced by his information η_k^i .

10. A prespecified class Γ_k^i (defined for $k \in K, i \in N$) of mappings $\gamma_k^i : N_k^i \rightarrow U_k^i$ which are the permissible strategies of \mathbf{Pi} at stage k of the game. The aggregate mapping $\gamma^i = \{\gamma_1^i, \dots, \gamma_T^i\}$ is a strategy of \mathbf{Pi} in the game. Furthermore the class Γ^i of all mappings γ^i is the strategy set (space) of \mathbf{Pi} .

11. A functional $L^i : (X \times U_1^1 \times \dots \times U_1^n) \times (X \times U_2^1 \times \dots \times U_2^n) \times \dots \times (X \times U_T^1 \times \dots \times U_T^n) \rightarrow \mathbf{R}$ (defined for $i \in N$) called the cost functional of \mathbf{Pi} in the game of fixed duration.

Definition 2 In an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)), P_i 's ($i \in N$) information structure is called $a(n)$

1. open-loop (OL) pattern if $\eta_k^i = \{x_0\}, (k \in K)$,
2. closed-loop perfect state information (CLPS) pattern if $\eta_k^i = \{x_0, \dots, x_{k-1}\}, (k \in K)$,
3. closed-loop imperfect state information (CLIS) pattern if $\eta_k^i = \{y_1^i, \dots, y_k^i\}, (k \in K)$,
4. memoryless perfect state information (MPS) pattern if $\eta_k^i = \{x_0, x_{k-1}\}, (k \in K)$,
5. feedback (perfect state) information (FB) pattern if $\eta_k^i = \{x_{k-1}\}, (k \in K)$,

Definition 3 For an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)), P_i 's ($i \in N$) cost functional is said to be stage-additive if there exist $g_k^i : X \times X \times U_k^1 \times \dots \times U_k^n \rightarrow \mathbf{R}, (k \in K)$, so that ($i, j \in N$)

$$L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}), \text{ where } u^j \triangleq (u_1^j, \dots, u_T^j).$$

Furthermore, if $L^i(u^1, \dots, u^n)$ depends only on x_T (the terminal state), then it is called terminal cost function.

Definition 4 An n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) is of affine-quadratic type if

$$f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k$$

$$L^i(u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$$

$$\begin{aligned} g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) &= \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) \\ &+ \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j \end{aligned}$$

where $U_k^i = \mathbf{R}^{m_i}$ and $s_k \in \mathbf{R}^p$. $A_k, B_k^i, Q_k^i, R_k^{ij}, \tilde{u}_k^{ij}, \tilde{x}_k^i$ (defined for $k \in K, i \in N, j \in N$) are fixed sequences of matrices or vectors of appropriate dimensions. Furthermore Q_k^i and R_k^{ij} are symmetric. An affine-quadratic game is of the linear-quadratic type if $s_k \equiv 0$.

Remark 1 The cost function of player i at stage k $[g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})]$ can also be written in the following way

$$\frac{1}{2}((x_k' - \tilde{x}_k^{i'}) Q_k^i (x_k - \tilde{x}_k^i) + \sum_{j \in N} (u_k^{j'} - \tilde{u}_k^{ij'}) R_k^{ij} (u_k^j - \tilde{u}_k^{ij'}))$$

Therefore $\tilde{x}_k^{i'}$ and $\tilde{u}_k^{ij'}$ can be interpreted as desired (target) values of each player for all variables of the game.

2.1.2 Solution concepts

In this subsection the Nash and Stackelberg equilibrium solution concepts are introduced for an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration. The Nash equilibrium solution concept provides a reasonable noncooperative equilibrium solution when no single player dominates the decision making process and therefore the roles of the players are symmetric. However, there are other types of noncooperative decision problems in which one of the players is a so-called leader and has the ability to enforce his strategy on the other players, the so-called followers. For that kind of decision problems a hierarchical equilibrium solution concept, the Stackelberg equilibrium solution, is introduced.

Definition 5 *An n -tuple of strategies $\{\gamma^{1*}, \gamma^{2*}, \dots, \gamma^{n*}\}$ with $\gamma^{i*} \in \Gamma^i, i \in N$, is said to constitute a noncooperative Nash equilibrium solution for an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) if the following n inequalities are satisfied for all $\gamma^i \in \Gamma^i, i \in N$:*

$$L^{1*} \triangleq L^1(\gamma^{1*}, \gamma^{2*}, \gamma^{3*}, \dots, \gamma^{n*}) \leq L^1(\gamma^1, \gamma^{2*}, \gamma^{3*}, \dots, \gamma^{n*})$$

$$L^{2*} \triangleq L^2(\gamma^{1*}, \gamma^{2*}, \gamma^{3*}, \dots, \gamma^{n*}) \leq L^2(\gamma^{1*}, \gamma^2, \gamma^{3*}, \dots, \gamma^{n*})$$

...

...

$$L^{n*} \triangleq L^n(\gamma^{1*}, \gamma^{2*}, \dots, \gamma^{n-1*}, \gamma^{n*}) \leq L^n(\gamma^{1*}, \gamma^2, \dots, \gamma^{n-1*}, \gamma^n)$$

$$\text{where } \gamma^i \triangleq (\gamma_1^i, \dots, \gamma_T^i)$$

The n -tuple of quantities $\{L^{1}, \dots, L^{n*}\}$ is known as a Nash equilibrium outcome of the discrete-time deterministic infinite dynamic game of prespecified fixed duration.*

Definition 6 In an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with $\mathbf{P1}$ as the leader the unique element $r^i(\gamma^1) \in \Gamma^i (i \in N, i \neq 1)$ defined for each $\gamma^1 \in \Gamma^1$ by

$$r^i(\gamma^1) = \min_{\gamma^i \in \Gamma^i} L^i(\gamma^1, r^2(\gamma^1), \dots, r^{i-1}(\gamma^1), \gamma^i, r^{i+1}(\gamma^1), \dots, r^n(\gamma^1))$$

where $\gamma^l \triangleq (\gamma_1^l, \dots, \gamma_T^l)$ and $r^l \triangleq (r_1^l, \dots, r_T^l)$; $l \in N$

is called the unique optimal response (rational reaction) of \mathbf{Pi} to the strategy $\gamma^1 \in \Gamma^1$ of $\mathbf{P1}$.

Definition 7 In an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with $\mathbf{P1}$ as the leader, a strategy $\gamma^{1*} \in \Gamma^1$ is called a Stackelberg equilibrium strategy for the leader if

$$L^{1*} \triangleq L^1(\gamma^{1*}, r^2(\gamma^{1*}), \dots, r^n(\gamma^{1*})) \leq L^1(\gamma^1, r^2(\gamma^1), \dots, r^n(\gamma^1)),$$

$$\forall \gamma^1 \in \Gamma^1, \forall r^i(\cdot) \in R^i(\cdot), i \in N, i \neq 1\}$$

where $\gamma^l \triangleq (\gamma_1^l, \dots, \gamma_T^l)$ and $r^l \triangleq (r_1^l, \dots, r_T^l)$

The quantity L^{1*} is called the Stackelberg cost of the leader.

Definition 8 In an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with $\mathbf{P1}$ as the leader, the element $\gamma^{i*} \in R^i(\gamma^{1*}) (i \in N, i \neq 1)$ is called an unique optimal strategy for follower i that is in equilibrium with γ^{1*} . The n -tuple $(\gamma^{1*}, \gamma^{2*}, \dots, \gamma^{n*})$ is a Stackelberg solution for the n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration with $\mathbf{P1}$ as the leader, and the n -tuple of quantities $\{L^{1*}, L^{2*}, \dots, L^{n*}\}$ is known as the corresponding Stackelberg equilibrium outcome.

2.1.3 Time consistency

The issue of time consistency has pervaded the economics literature during the past three decades, following the important paper by Kydland and Prescott (1977) [10]. Based on Başar's paper (1989) [1] we impose further refinements on the class of equilibrium strategies drawing a distinction between time inconsistent, weakly and strongly time consistent optimal strategies that will be used later on to characterize the quality of different equilibrium solution concepts under different information patterns.

Definition 9 *An n -tuple of policies (equilibrium strategies) $(\gamma^{1*}, \dots, \gamma^{n*}) \in \Gamma$ solving the dynamic game defined in Definition (1) for any particular information pattern defined in Definition (2) is weakly time consistent (WTC) if its truncation of stages to the interval $[s, T]$ (for $s \in [1, \dots, T]$), $(\gamma_{[s, T]}^{1*}, \dots, \gamma_{[s, T]}^{n*})$ solves the truncated game. If an n -tuple of policies $(\gamma^{1*}, \dots, \gamma^{n*}) \in \Gamma$ is not WTC, then it is called time inconsistent.*

Definition 10 *An n -tuple of policies (optimal strategies) $(\gamma^{1*}, \dots, \gamma^{n*}) \in \Gamma$ solving the dynamic game defined in Definition (1) for any particular information pattern defined in Definition (2) is strongly time consistent (STC) if its truncation of stages to the interval $[s, T]$ (for $s \in [1, \dots, T]$), $(\gamma_{[s, T]}^{1*}, \dots, \gamma_{[s, T]}^{n*})$ solves the truncated game for every permissible n -tuple of strategies played in the interval $[0, s)$ $l \triangleq (\gamma_{[0, s)}^1, \dots, \gamma_{[0, s)}^n) \in \Gamma$.*

2.2 Basic insights

This section presents the most important optimization tools for discrete-time dynamic noncooperative game theory and some results about matrix identities and the definiteness of matrices that will be used throughout the next chapters.

2.2.1 Dynamic programming

The method of *dynamic programming* was developed by Richard Bellman (cf. Bellman (1957) [3]) and is a tool for solving games with feedback information pattern. It is based on the *principle of optimality*, which states that an optimal strategy has the property that, whatever the initial state and time are, all remaining decisions (from that particular initial state and time onwards) must also constitute an optimal strategy. To make use of this principle in a mathematical dynamic (game theoretic) framework, we have to work backwards in time, starting at all possible final states with the corresponding final times.

To apply the *principle of optimality* to our game theoretic framework (cf. Def. (1)), we have to consider a stage-additive cost functional (cf. Def. (3)) and feedback (perfect state) information pattern (cf. Def. (2)):

$$L^i(u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}),$$

$$\text{where } u^j \triangleq (u_1^j, \dots, u_T^j) \text{ and } u_k^i = \gamma_k^i(x_{k-1}); j \in N$$

$$x_k = f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n), k \in K$$

On this basis we can define an expression for the minimal cost of $\mathbf{P}i$ ($i \in N$) for any starting point and any corresponding initial time.

Definition 11 For an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)), \mathbf{P}^i 's ($i \in N$) value function is defined as:

$$V^i(k, x_{k-1}) = \min_{u_k^i \dots u_T^i} \sum_{j=k}^T g_j^i(x_j, u_j^1, \dots, u_j^n, x_{j-1})$$

Because of the principle of optimality, the value function is equivalent to the following recursive relation:

$$\begin{aligned} V^i(k, x_{k-1}) = \min_{u_k^i \in U_k^i} [& g_k^i(f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n), u_k^1, \dots, u_k^n, x_{k-1}) \\ & + V^i(k+1, f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n))] \end{aligned}$$

Solving dynamic problems by using this recursive relation is known as *dynamic programming*.

2.2.2 The minimum principle

The *minimum principle* was developed (in continuous time) by Lew Semjonowitsch Pontryagin (cf. Pontryagin et al. (1962) [12]) and is a tool for solving games with open-loop (and feedback) information pattern. In the version presented below sufficient conditions for the existence of optimal solutions are given because the following Theorem (1) will be applied (via Theorem (6)) to affine-quadratic games which fulfill the stated sufficient conditions. The derivation of Theorem (1) can be found in either Canon et al. (1970) [5] or Boltyanski (1978) [4].

Theorem 1 *For the discrete-time optimal control problem (cf. Def. (1) with $N = \{1\}$) let*

- $f_k(\cdot, \cdot)$ be continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^m$ (defined for: $k \in K$)
- $g_k(\cdot, \cdot, \cdot)$ be continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^m \times \mathbf{R}^p$ (defined for: $k \in K$)
- $f_k(\cdot, \cdot)$ be convex on $\mathbf{R}^p \times \mathbf{R}^m$ (defined for: $k \in K$)
- $g_k(\cdot, \cdot, \cdot)$ be convex on $\mathbf{R}^p \times \mathbf{R}^m \times \mathbf{R}^p$ (defined for: $k \in K$)
- the cost function be stage-additive (cf. Def. (3)).

Then $\{\gamma_k^*(x_0) = u_k^*\}$ denotes an optimal control sequence and $\{x_k^*; k \in K\}$ is the corresponding state trajectory and there exists a finite sequence of p -dimensional costate vectors $\{p_1, \dots, p_T\}$ so that the following relations are satisfied:

$$(2.1) \quad x_k^* = f_{k-1}(x_{k-1}^*, u_k^*), \quad x_0^* = x_0$$

$$(2.2) \quad \nabla_{u_k} H_k(p_k, u_k^*, x_{k-1}^*) = 0$$

$$(2.3) \quad p_k = \frac{\partial}{\partial x_k} f_k(x_k^*, u_{k+1}^*)' [p_{k+1} + (\frac{\partial}{\partial x_{k+1}} g_{k+1}(x_{k+1}^*, u_{k+1}^*, x_k^*))'] \\ + [\frac{\partial}{\partial x_k} g_{k+1}(x_{k+1}^*, u_{k+1}^*, x_k^*)]'; \quad p_T = 0$$

where

$$(2.4) \quad H_k(p_k, u_k, x_{k-1}) \hat{=} g_k(f_{k-1}(x_{k-1}, u_k), u_k, x_{k-1}) + p_k' f_{k-1}(x_{k-1}, u_k)$$

2.2.3 Some results about the definiteness of matrices

This subsection is devoted to the derivation of Corollary (1), which is needed for convexity analyses later on.

Lemma 1 $A + B > 0$ if

- $A > 0, B \geq 0$.

PROOF:

$$(2.5) \quad x'(A + B)x = x'Ax + x'Bx > 0 \quad \square$$

Lemma 2 $B'AB \geq 0$ if

- $A \geq 0$.

PROOF:

For showing

$$(2.6) \quad x'B'ABx \geq 0 \quad \forall x$$

we define a linear mapping L

$$(2.7) \quad L: \tilde{x} = Bx \quad \forall x$$

If B has full rank, the vectors \tilde{x} cover the whole space and \tilde{x} is only zero if the corresponding vector x is also zero. If B has not full rank, some vectors \tilde{x} are

mapped to zero for vectors x unequal to zero. Therefore the vectors \tilde{x} cover only a part of the space. In both cases

$$(2.8) \quad \tilde{x}'A\tilde{x} \geq 0 \quad \forall \tilde{x}$$

holds true because $A \geq 0$. \square

Corollary 1 $A + C'BC > 0$ if

- $A > 0, B \geq 0$.

PROOF:

$D := C'BC$ has to be ≥ 0 because of Lemma (2). Now we can apply Lemma (1) to $A + D$. \square

2.2.4 Some results about matrix identities

In this subsection two matrix identities are deduced. These are needed later on in some propositions which relate my findings with the results stated in Başar and Olsder (1999) [2].

First a general result is given concerning the eigenvalues of a product of two matrices with certain properties, whose derivation can be found in Horn and Johnson (1991, p. 465)[9].

Lemma 3 *The product of a positive definite matrix A and a Hermitian matrix B is a diagonalizable matrix, all of whose eigenvalues are real. The matrix AB has the same number of positive, negative and zero eigenvalues as B .*

Now Lemma (3) is applied to deduce two matrix identities, which will be used several times in Proposition (4) and Proposition (6).

Lemma 4 *Let A be positive definite and B a be matrix of appropriate dimension, then the following two matrix identities hold:*

1. $[I - AB(I + B'AB)^{-1}B'] \equiv (I + ABB')^{-1}$
2. $[I - B(I + B'AB)^{-1}B'A] \equiv (I + BB'A)^{-1}$

PROOF:

We will prove matrix identity 1 and the proof of 2 can be done in essentially the same way.

At first note that if A is positive definite, $B'AB$ is positive semidefinite with Lemma (2) and ABB' is positive semidefinite with Lemma (3).³ Therefore the matrices $(I + B'AB)^{-1}$ and $(I + ABB')^{-1}$ exist.

³ Marcus and Minc (1964, p. 24) [11] show that both terms have the same eigenvalues.

Now we have to show that $[I - AB(I + B'AB)^{-1}B']$ multiplied by $(I + ABB')$ gives the identity matrix I .

$$[I - AB(I + B'AB)^{-1}B'](I + ABB')$$

Expanding and prescinding yields

$$I + ABB' - AB(I + B'AB)^{-1}B' - AB(I + B'AB)^{-1}B'ABB'$$

$$I + AB[I - (I + B'AB)^{-1} - (I + B'AB)^{-1}B'AB]B'$$

$$I + AB[I - (I + B'AB)^{-1}(I + B'AB)]B' = I + AB[I - I]B' = I \quad \square$$

3 Discrete-Time Infinite Dynamic Games with Feedback Information Pattern

3.1 Feedback Nash Equilibrium Solutions

This section is devoted to the derivation of the so-called feedback Nash equilibrium solution for affine-quadratic games. First a general result is stated about the existence and uniqueness of a feedback Nash equilibrium solution in n -person discrete-time deterministic infinite dynamic games of prespecified fixed duration (cf. Def. (1)) with feedback information pattern. Then this result is applied to affine-quadratic games and finally the solution for the affine-quadratic control problem is deduced as a special case.

3.1.1 Optimality conditions

In this subsection a theorem is stated which gives necessary and sufficient conditions for the existence of a feedback Nash equilibrium solution. Results about feedback Nash equilibria in infinite dynamic games first appeared in continuous time in the works of Starr and Ho (1969) [15], [16] and Case (1969) [6]. A proof of Theorem (2), which is the counterpart of the above-mentioned results in discrete time, can be found in Başar and Olsder (1999, pp. 278-279)[2].

Theorem 2 *For an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with feedback information pattern, the set of strategies $\{\gamma_k^{i*}(x_k); k \in K, i \in N\}$ provides a feedback Nash equilibrium solution if, and only if, functions $V^i(k, \cdot) : R^n \rightarrow R(k \in K, i \in N)$ exist such that the following*

recursive relations are satisfied:

$$\begin{aligned}
 (3.1) \quad V^i(k, x_{k-1}) = \min_{u_k^i \in U_k^i} [& g_k^i(\tilde{f}_{k-1}^{i*}(x_{k-1}, u_k^i), \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{j-1*}(x_{k-1}), u_k^i, \\
 & \gamma_k^{j+1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}), x_{k-1}) + V^i(k+1, \tilde{f}_{k-1}^{i*}(x_{k-1}, u_k^i))] = \\
 & g_k^i[\tilde{f}_{k-1}^{i*}(x_{k-1}, \gamma_k^{i*}(x_{k-1})), \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}), x_{k-1}] \\
 & + V^i[k+1, \tilde{f}_{k-1}^{i*}(x_{k-1}, \gamma_k^{i*}(x_{k-1}))]; \quad V^i(T+1, x_T) = 0
 \end{aligned}$$

where $\tilde{f}_{k-1}^{i*}(x_{k-1}, z) \triangleq$

$$f_{k-1}(x_{k-1}, \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{j-1*}(x_{k-1}), z, \gamma_k^{j+1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}))$$

Every such equilibrium solution is strongly time consistent, and the corresponding Nash equilibrium cost for \mathbf{P}^i is $V^i(1, x_0)$.

3.1.2 Results for affine-quadratic games with arbitrarily many players

In the following, the results of Theorem (2) are applied to an affine-quadratic dynamic game with arbitrarily many players. Theorem (3), which is an extension (concerning the cost functionals) of Corollary 6.1 in Başar and Olsder (1999, pp. 279-281)[2], presents equilibrium equations that can easily be used for an algorithmic disintegration of the given Nash game.

Furthermore in Proposition (1) the equivalence of the equations in Theorem (3) with terminologically different equations is shown.

Theorem 3 *An n -person affine-quadratic dynamic game (cf. Def. (4)) admits a unique feedback Nash equilibrium solution if*

- $Q_k^i \geq 0, R_k^{ii} > 0$ (defined for $k \in K, i \in N$).
- (3.6) and (3.7) admit unique optimal solution sets P_k^{i*} and α_k^{i*} (defined for: $k \in K, i \in N$)

If these conditions are satisfied, the unique equilibrium strategies are given by (3.5) and the corresponding feedback Nash equilibrium cost for each player is stated in (3.10).⁴

$$(3.2) \quad f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k$$

$$(3.3) \quad L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$$

$$(3.4) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) \\ + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j$$

⁴ For all equations belonging to this theorem and its proof, $i \in N$ and $k \in K$ if nothing different is stated.

$$(3.5) \quad \gamma_k^{i*}(x_{k-1}) = u_k^{i*} = -P_k^{i*}x_{k-1} - \alpha_k^{i*}$$

$$(3.6) \quad P_k^{i*} = (R_k^{ii} + B_k^{i'}Z_k^iB_k^i)^{-1}[B_k^{i'}Z_k^i(A_k - \sum_{j \in N, j \neq i} B_k^jP_k^{j*})]$$

$$(3.7) \quad \alpha_k^{i*} = (R_k^{ii} + B_k^{i'}Z_k^iB_k^i)^{-1}[B_k^{i'}(Z_k^i(s_k - \sum_{j \in N, j \neq i} B_k^j\alpha_k^{j*}) \\ + \zeta_k^i - Q_k^i\tilde{x}_k^i) - R_k^{ii}\tilde{u}_k^{ii}]$$

$$(3.8) \quad Z_{k-1}^i = (A_k - \sum_{j \in N} B_k^jP_k^{j*})'Z_k^i(A_k - \sum_{j \in N} B_k^jP_k^{j*}) \\ + \sum_{j \in N} P_k^{j*'}R_k^{ij}P_k^{j*} + Q_{k-1}^i; Z_T^i = Q_T^i$$

$$(3.9) \quad \zeta_{k-1}^i = (A_k - \sum_{j \in N} B_k^jP_k^{j*})'[\zeta_k^i + Z_k^i(s_k - \sum_{j \in N} B_k^j\alpha_k^{j*}) - Q_k^i\tilde{x}_k^i] \\ + \sum_{j \in N} P_k^{j*'}R_k^{ij}(\alpha_k^{j*} + \tilde{u}_k^{ij}); \zeta_T^i = 0$$

$$(3.10) \quad V^i(1, x_0) = \frac{1}{2}x_0'Z_0^ix_0 + \zeta_0^{i'}x_0 + n_0^i$$

$$(3.11) \quad n_{k-1}^i = n_k^i + \frac{1}{2}(s_k - \sum_{j \in N} B_k^j\alpha_k^{j*})'Z_k^i(s_k - \sum_{j \in N} B_k^j\alpha_k^{j*}) \\ + \zeta_k^{i'}(s_k - \sum_{j \in N} B_k^j\alpha_k^{j*}) + \frac{1}{2}\sum_{j \in N} \alpha_k^{j*'}R_k^{ij}\alpha_k^{j*} - \tilde{x}_k^{i'}Q_k^i(s_k - \sum_{j \in N} B_k^j\alpha_k^{j*}) \\ + \sum_{j \in N} \tilde{u}_k^{ij'}R_k^{ij}\alpha_k^{j*} + \frac{1}{2}(\tilde{x}_k^{i'}Q_k^i\tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'}R_k^{ij}\tilde{u}_k^{ij}); n_T^i = 0$$

PROOF:

The proof is carried out by using an induction argument to show that the strategies $\gamma_k^{i*}(x_{k-1})$, given by (3.5), minimize the strictly convex functionals (3.12) at each stage of the game. But the minimization of (3.12) is exactly (3.1) applied to the specific state equation and cost functionals of the above game and therefore, considering Theorem (2), it follows that the $\gamma_k^{i*}(x_{k-1})$ are the unique equilibrium strategies.

$$\begin{aligned}
 (3.12) \quad & \frac{1}{2}((A_k x_{k-1} + B_k^i u_k^i + \sum_{j \in N, j \neq i} B_k^j u_k^{j*} + s_k)' Q_k^i (A_k x_{k-1} + B_k^i u_k^i \\
 & + \sum_{j \in N, j \neq i} B_k^j u_k^{j*} + s_k) + u_k^{i'} R_k^{ii} u_k^i + \sum_{j \in N, j \neq i} u_k^{j*'} R_k^{ij} u_k^{j*}) + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i \\
 & + \sum_{j \in N, j \neq i} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i (A_k x_{k-1} + B_k^i u_k^i + \sum_{j \in N, j \neq i} B_k^j u_k^{j*} + s_k) - \tilde{u}_k^{ii'} R_k^{ii} u_k^i \\
 & - \sum_{j \in N, j \neq i} \tilde{u}_k^{ij'} R_k^{ij} u_k^{j*} + V^i(k+1, \tilde{f}_{k-1}^{i*}(x_{k-1}, u_k^i)) ; V^i(T+1, x_T) = 0
 \end{aligned}$$

The induction argument runs from $T+1$ to 1 and proves that the value function of player i at stage k can be written as stated below in (3.13):

$$(3.13) \quad \frac{1}{2} x_{k-1}' (Z_{k-1}^i - Q_{k-1}^i) x_{k-1} + \zeta_{k-1}^{i'} x_{k-1} + n_{k-1}^i$$

Basis:

The induction starts at $k = T + 1$. First we make use of the general optimality conditions for V_k^i at stage $T+1$.

$$(3.14) \quad V^i(T+1, x_T) = 0$$

Now we show that $(3.13)_{k=T+1}$ fulfills (3.14).

$$(3.13)_{k=T+1} \quad \frac{1}{2}x_T'(Z_T^i - Q_T^i)x_T + \zeta_T^{i'}x_T + n_T^i$$

Making use of $(3.8)_{k=T+1}$, $(3.9)_{k=T+1}$ and $(3.11)_{k=T+1}$ yields

$$\frac{1}{2}x_T'(Q_T^i - Q_T^i)x_T + 0'x_T + 0 = 0 = V^i(T+1, x_T)$$

Inductive step:

As an induction hypothesis, the equations $(3.13)_{k=l+1}$ are assumed to be equal to $V^i(l+1, x_l)$ respectively. Now we have to prove that the relation also holds at stage l . In other words: we have to show $(3.13)_{k=l} = V^i(l, x_{l-1})$.

$$(3.13)_{k=l+1} \quad \frac{1}{2}x_l'(Z_l^i - Q_l^i)x_l + \zeta_l^{i'}x_l + n_l^i$$

Using $(3.2)_{k=l}$ and considering that the optimal control vector for player i remains to be deduced in the further induction argument, gives

$$(3.15) \quad \frac{1}{2}(A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l)'(Z_l^i - Q_l^i)(A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) + \zeta_l^{i'}(A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) + n_l^i$$

First we prove that $(3.5)_{k=l}$ minimizes $(3.12)_{k=l}$. To do this, we substitute $V^i(l+1, \tilde{f}_{l-1}^{i*}(x_{l-1}, u_l^i))$ in $(3.12)_{k=l}$ with the help of the induction hypothesis in (3.15):

$$\begin{aligned}
 (3.12)_{k=l} &= \frac{1}{2}((A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l)' Q_l^i (A_l x_{l-1} + B_l^i u_l^i \\
 &\quad + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) + u_l^{i'} R_l^{ii} u_l^i + \sum_{j \in N, j \neq i} u_l^{j*'} R_l^{ij} u_l^{j*}) + \frac{1}{2}(\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i \\
 &\quad + \sum_{j \in N, j \neq i} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) - \tilde{u}_l^{ii'} R_l^{ii} u_l^i \\
 &\quad - \sum_{j \in N, j \neq i} \tilde{u}_l^{ij'} R_l^{ij} u_l^{j*} + V^i(l+1, \tilde{f}_{l-1}^{i*}(x_{l-1}, u_l^i)) \\
 (3.16) &= \frac{1}{2}((A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l)' Q_l^i (A_l x_{l-1} + B_l^i u_l^i \\
 &\quad + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) + u_l^{i'} R_l^{ii} u_l^i + \sum_{j \in N, j \neq i} u_l^{j*'} R_l^{ij} u_l^{j*}) + \\
 &\quad \frac{1}{2}(\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) \\
 &\quad - \tilde{u}_l^{ii'} R_l^{ii} u_l^i - \sum_{j \in N, j \neq i} \tilde{u}_l^{ij'} R_l^{ij} u_l^{j*} + \frac{1}{2}(A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l)' (Z_l^i - Q_l^i) \\
 &\quad (A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) + \zeta_l^{i'} (A_l x_{l-1} + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j u_l^{j*} + s_l) + n_l^i
 \end{aligned}$$

$(3.12)_{k=l}$ is strictly convex in u_l^i . This can be seen by applying Corollary (1) to (3.17). Therefore there has to be a unique equilibrium strategy for player i at stage l .

$$(3.17) \quad \frac{\partial^2}{\partial u_l^{i2}} (3.16) \quad R_l^{ii} + B_l^{i'} Z_l^i B_l^i$$

This unique optimal strategy can be found by using the first-order necessary and

sufficient (because of strict convexity of $(3.12)_{k=l}$) conditions for minimization

$$\begin{aligned} \frac{\partial}{\partial u_l^i}(3.16) = 0 \Rightarrow & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i) u_l^{i*} - B_l^{i'} Z_l^i \\ & \sum_{j \in N, j \neq i} B_l^j u_l^{j*} = B_l^{i'} [Z_l^i (A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i] - R_l^{ii} \tilde{u}_l^{ii} \end{aligned}$$

As the right hand side of the above equation is affine in x_{l-1} , the left hand side also has to be affine in x_{l-1} . Therefore, the substitution

$$(3.18) \quad u_l^{i*} = -P_l^{i*} x_{l-1} - \alpha_l^{i*}$$

is allowed and leads to

$$\begin{aligned} (3.19) \quad & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i) (-P_l^{i*} x_{l-1} - \alpha_l^{i*}) - B_l^{i'} Z_l^i \\ & \sum_{j \in N, j \neq i} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) = B_l^{i'} [Z_l^i (A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i] - R_l^{ii} \tilde{u}_l^{ii} \end{aligned}$$

By comparison of coefficients follows

$$(3.19)_{x_{l-1}} \quad (R_l^{ii} + B_l^{i'} Z_l^i B_l^i) P_l^{i*} + B_l^{i'} Z_l^i \sum_{j \in N, j \neq i} B_l^j P_l^{j*} = B_l^{i'} Z_l^i A_l$$

$$(3.19)_{const.} \quad (R_l^{ii} + B_l^{i'} Z_l^i B_l^i) \alpha_l^{i*} + B_l^{i'} Z_l^i \sum_{j \in N, j \neq i} B_l^j \alpha_l^{j*} = B_l^{i'} [Z_l^i s_l + \zeta_l^i - Q_l^i \tilde{x}_l^i] - R_l^{ii} \tilde{u}_l^{ii}$$

Making P_l^{i*} and α_l^{i*} explicit yields

$$(3.6)_{k=l} \quad P_l^{i*} = (R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} Z_l^i (A_l - \sum_{j \in N, j \neq i} B_l^j P_l^{j*})]$$

$$(3.7)_{k=l} \quad \alpha_l^{i*} = (R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (s_l - \sum_{j \in N, j \neq i} B_l^j \alpha_l^{j*}) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}]$$

Now, after finding the optimal strategies for the players, we are able to rewrite (3.16) as

$$\begin{aligned} V^i(l, x_{l-1}) = & \frac{1}{2} [(A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l)' Z_l^i (A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l) \\ & + \sum_{j \in N} u_l^{j*'} R_l^{ij} u_l^{j*}] + \frac{1}{2} (\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l) \\ & - \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} u_l^{j*} + \zeta_l^{i'} (A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l) + n_l \end{aligned}$$

Making use of (3.18) yields

$$\begin{aligned} V^i(l, x_{l-1}) = & \frac{1}{2} [(A_l x_{l-1} + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l)' Z_l^i (A_l x_{l-1} \\ & + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l) + \sum_{j \in N} (-P_l^{j*} x_{l-1} - \alpha_l^{j*})' R_l^{ij} (-P_l^{j*} x_{l-1} - \alpha_l^{j*})] \\ & + \frac{1}{2} (\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l) \\ & - \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + \zeta_l^{i'} (A_l x_{l-1} + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l) + n_l \end{aligned}$$

Rewriting the above equation to the power of x_{l-1} gives

$$\begin{aligned}
 V^i(l, x_{l-1}) = & \frac{1}{2} x'_{l-1} [(A_l - \sum_{j \in N} B_l^j P_l^{j*})' Q_l^i (A_l - \sum_{j \in N} B_l^j P_l^{j*}) + \sum_{j \in N} P_l^{j*'} R_l^{ij} P_l^{j*}] x_{l-1} \\
 & + [(A_l - \sum_{j \in N} B_l^j P_l^{j*})' [\zeta_l^i Z_l^i (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) - Q_l^i \tilde{x}_k^i] + \sum_{j \in N} P_l^{j*'} R_l^{ij} (\alpha_l^{j*} + \tilde{u}_l^{ij})]' x_{l-1} \\
 & + \frac{1}{2} (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*})' Z_l^i (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) + \frac{1}{2} \sum_{j \in N} \alpha_l^{j*'} R_l^{ij} \alpha_l^{j*} \\
 & - \tilde{x}_l^{i'} Q_l^i (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \alpha_l^{j*} + \frac{1}{2} (\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) \\
 & + \zeta_l^{i'} (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) + n_l
 \end{aligned}$$

To finish off the inductive step and consequently the induction argument, we use the recursive equations (3.8) $_{k=l}$, (3.9) $_{k=l}$ and (3.11) $_{k=l}$ in the above equation. This leads to

$$(3.13)_{k=l} \quad V^i(l, x_{l-1}) = \frac{1}{2} x'_{l-1} (Z_{l-1}^i - Q_{l-1}^i) x_{l-1} + \zeta_{l-1}^{i'} x_{l-1} + n_{l-1}^i$$

The expression for the total costs of the game for player i given by (3.10) is equal to the function the induction argument was based on at stage l . In other words, (3.10) is equal to (3.13) $_{k=1}$. \square

Remark 2 *The proof of Theorem (3) is a formalization of the heuristic argumentation presented in Başar and Olsder (1999, pp. 280-281)[2].*

Remark 3 Z_k^i , given by (3.8), is positive definite for all $k \in \{0, \dots, T\}$. This can be proven by a straightforward induction argument, starting at stage T and using Corollary (1) in the inductive step.

Remark 4 *Special attention should be paid to the observation that using an affine state equation together with a quadratic cost functional yields affine equilibrium strategies. This also holds true for feedback Stackelberg, open-loop Nash and open-loop Stackelberg games.*

Remark 5 To solve the Nash game algorithmically, the following order of application of the equations of Theorem (3) is advisable ($i \in N$):

1. For k running backward from T to 1

a) Z_k^i, ζ_k^i and n_k^i

b) P_k^{i*}, α_k^{i*}

2. Z_0^i, ζ_0^i and n_0^i

3. $V^i(1, x_0)$

4. For k running forward from 1 to T

$\gamma_k^{i*}(x_{k-1})$

Proposition 1⁵ The systems of equations defining the unique equilibrium strategies $\gamma_k^{i*}(x_{k-1})$ in Theorem (3) can also be written in the following way:⁶

$$(3.20) \quad \gamma_k^{i*}(x_{k-1}) = G_k^{i*} x_{k-1} + g_k^{i*}$$

$$(3.21) \quad G_k^{i*} = -(D_k^i)^{-1} [B_k^{i'} H_k^i (A_k + \sum_{j \in N, j \neq i} B_k^j G_k^{j*})]$$

$$(3.22) \quad g_k^{i*} = -(D_k^i)^{-1} [B_k^{i'} H_k^i \sum_{j \in N, j \neq i} B_k^j g_k^{j*} + v_k^i]$$

$$(3.23) \quad H_{k-1}^i = K_k^j Z_k^i K_k + \sum_{j \in N} G_k^{j'} R_k^{ij} G_k^j + Q_{k-1}^i; H_T^i = Q_T^i$$

$$(3.24) \quad h_{k-1}^i = Q_{k-1}^i \tilde{x}_{k-1}^i - K_k^j [H_k^i k_k - h_{k+1}^i] + \sum_{j \in N} G_k^{j'} R_k^{ij} (\tilde{u}_k^{ij} - g_k^j); h_T^i = Q_T^i \tilde{x}_T^i$$

⁵ In this proposition we rewrite the equilibrium equations in a notation that was used at our department in the past to enable comparison.

⁶ For all equations belonging to this proposition and its proof, $k \in K$ and $i \in N$ if nothing different is stated.

$$(3.25) \quad K_k = A_k + \sum_{j \in N} B_k^i G_k^j$$

$$(3.26) \quad k_k = s_k + \sum_{j \in N} B_k^j g_k^j$$

$$(3.27) \quad D_k^i = R_k^{ii} + B_k^{i'} H_k^i B_k^i$$

$$(3.28) \quad v_k^i = B_k^{i'} (H_k^i s_k - h_k^i) - R_k^{ii} \tilde{u}_k^{ii}; i \in N$$

PROOF:

The proof is carried out by renaming some matrices and then showing that the relations for the equilibrium strategies $\gamma_k^{i*}(x_{k-1})$ of Theorem (5) can be rewritten in the way stated above.

Let us start by renaming the feedback matrices P_k^{i*} and α_k^{i*} and the matrices Z_k^i and ζ_{k+1}^i .

$$(3.29) \quad P_k^i \triangleq -G_k^i; \alpha_k^i \triangleq -g_k^i; Z_k^i \triangleq H_k^i; \zeta_k^i \triangleq -h_k^i + Q_k^i \tilde{x}_k^i$$

Next we prove that Z_k^i and ζ_k^i fulfill (3.23) and (3.24) respectively.

Taking the renaming (3.8) into account gives

$$H_{k-1}^i = (A_k + \sum_{j \in N} B_k^i G_k^j)' H_k^i (A_k + \sum_{j \in N} B_k^j G_k^j) + \sum_{j \in N} G_k^{j'} R_k^{ij} G_k^j + Q_{k-1}^i$$

Making use of (3.25) yields

$$(3.23) \quad H_{k-1}^i = K_k' H_k^i K_k + \sum_{j \in N} G_k^{j'} R_k^{ij} G_k^j + Q_{k-1}^i$$

Now we show the correctness of equation (3.24). To do so we start with stage T and use (3.9) and (3.29) to get

$$0 = \zeta_T^i = -h_T^i + Q_T^i \tilde{x}_T^i$$

$$(3.24)_{k=T} \quad h_T^i = Q_T^i \tilde{x}_T^i$$

For the general stage k rewrite (3.9) taking consideration of (3.29)

$$-h_{k-1}^i + Q_{k-1}^i \tilde{x}_{k-1}^i = (A_k + \sum_{j \in N} B_k^j G_k^j)' [-h_k^i + H_k^i (s_k + \sum_{j \in N} B_k^j g_k^j)] - \sum_{j \in N} G_k^{j'} R_k^{ij} (\tilde{u}_k^{ij} - g_k^j)$$

Using (3.25) and (3.26) and making h_{k-1}^i explicit yields

$$(3.24) \quad h_{k-1}^i = Q_{k-1}^i \tilde{x}_{k-1}^i - K_k' [H_k^i k_k - h_k^i] + \sum_{j \in N} G_k^{j'} R_k^{ij} (\tilde{u}_k^{ij} - g_k^j)$$

Eventually the correctness of the rewritten equilibrium strategies $\gamma_k^{i*}(x_{k-1})$ given by (3.20) - (3.22) has to be shown.

First substituting P_k^{i*} and α_k^{i*} in (3.5) with the help of (3.29) leads to

$$(3.20) \quad \gamma_k^{i*}(x_{k-1}) = G_k^{i*} x_{k-1} + g_k^{i*}$$

Taking (3.29) into consideration, the feedback matrices given by (3.6) and (3.7) can be rewritten as

$$G_k^{i*} = -(R_k^{ii} + B_k^{i'} H_k^i B_k^i)^{-1} [B_k^{i'} H_k^i (A_k + \sum_{j \in N, j \neq i} B_k^j G_k^{j*})]$$

$$(3.30) \quad g_k^{i*} = -(R_k^{ii} + B_k^{i'} H_k^i B_k^i)^{-1} [B_k^{i'} (H_k^i (s_k + \sum_{j \in N, j \neq i} B_k^j g_k^{j*}) - h_k^i) - R_k^{ii} \tilde{u}_k^{ii}]$$

Finally using (3.27) in the two equations above and additionally using (3.28) in (3.30) gives

$$(3.21) \quad G_k^{i*} = -(D_k^i)^{-1} [B_k^{i'} H_k^i (A_k + \sum_{j \in N, j \neq i} B_k^j G_k^{j*})]$$

$$(3.22) \quad g_k^{i*} = -(D_k^i)^{-1} [B_k^{i'} H_k^i \sum_{j \in N, j \neq i} B_k^j g_k^{j*} + v_k^i] \quad \square$$

3.1.3 Special case: The affine-quadratic control problem

In this subsection the results of Theorem (3) are first specialized in Corollary (2) by reducing the number of players from n to one and then in Proposition (2) the specialized results are transformed into the terminology used in Proposition 5.1 in Başar and Olsder (1999, pp. 234-235)[2] to point out some serious mistakes stated there.

Corollary 2 *An affine-quadratic control problem (cf. Def. (1) with $N = \{1\}$ and (3.31) - (3.33)) admits the unique control solution if*

- $Q_k \geq 0, R_k > 0$ (defined for: $k \in K$).

If these conditions are satisfied, the unique optimal strategies are given by (3.34). The corresponding minimum value is stated in (3.39).⁷

$$(3.31) \quad f_k(x_k, u_k) = A_k x_k + B_k u_k + c_k$$

$$(3.32) \quad L(u) = \sum_{k=1}^T g_k(x_{k+1}, u_k, x_k)$$

$$(3.33) \quad g_k(x_{k+1}, u_k, x_k) = \frac{1}{2}(x'_{k+1} Q_{k+1} x_{k+1} + u'_k R_k u_k)$$

$$(3.34) \quad \gamma_k(x_k) = -P_k^* x_k - \alpha_k^*$$

$$(3.35) \quad P_k^* = (R_k + B_k Z_{k+1} B_k)^{-1} B'_k Z_{k+1} A_k$$

$$(3.36) \quad \alpha_k^* = (R_k + B'_k Z_{k+1} B_k)^{-1} B'_k (Z_{k+1} c_k + \zeta_{k+1})$$

$$(3.37) \quad Z_k = (A_k - B_k P_k^*)' Z_{k+1} (A_k - B_k P_k^*) + P_k^{*'} R_k P_k^* + Q_k ; Z_T = Q_T$$

⁷ For all equations belonging to this corollary, its proof and its equivalence analysis, $k \in K$ if nothing different is stated.

$$(3.38) \quad \zeta_k = (A_k - B_k P_k^*)' [\zeta_{k+1} + Z_{k+1} (c_k - B_k \alpha_k^*)] + P_k^{*'} R_k \alpha_k^* ; \zeta_{T+1} = 0$$

$$(3.39) \quad V(1, x_1) = \frac{1}{2} x_1' Z_1 x_1 + \zeta_1' x_1 + n_1$$

$$(3.40) \quad n_k = n_{k+1} + \frac{1}{2} (c_k - B_k \alpha_k^*)' Z_{k+1} (c_k - B_k \alpha_k^*) \\ + \zeta_{k+1}' (c_k - B_k \alpha_k^*) + \frac{1}{2} \alpha_k^{*'} R_k \alpha_k^*$$

PROOF:

Corollary (2) is proven in the same way as Theorem (3) taking into consideration simplifications resulting from the reduction in the number of players to one and the modified state equation and cost functionals. \square

Proposition 2 *The systems of equations defining the unique equilibrium strategies $\gamma_{k+1}^*(x_k)$ in Corollary (2) can also be written in the following way:*⁸

$$(3.41) \quad \gamma_k^*(x_k) = -P_k S_{k+1} A_k x_k - P_k (s_{k+1} + S_{k+1} c_k)$$

$$(3.42) \quad P_k = (R_k + B_k' S_{k+1} B_k)^{-1} B_k'$$

$$(3.43) \quad S_k = (A_k - B_k P_k^*)' S_{k+1} (A_k - B_k P_k^*) + P_k^{*'} R_k P_k^* + Q_k ; S_{T+1} = Q_{T+1}$$

$$(3.44) \quad s_k = (A_k - B_k P_k^*)' [s_{k+1} + S_{k+1} (c_k - B_k \alpha_k^*)] + P_k^{*'} R_k \alpha_k^* ; s_{T+1} = 0$$

$$(3.45) \quad L(1, x_1) = \frac{1}{2} x_1' S_1 x_1 + s_1' x_1 + q_1$$

⁸ For all equations belonging to this proposition and its proof, $k \in K$ and $i \in N$ if nothing different is stated. Equations (3.43), (3.44) and (3.46) are wrong in Başar and Olsder.

$$(3.46) \quad q_k = q_{k+1} + \frac{1}{2}(c_k - B_k P_k(S_{k+1}c_k + s_{k+1}))' S_{k+1}(c_k - B_k P_k(S_{k+1}c_k + s_{k+1})) \\ + s_{k+1}'(c_k - B_k P_k(S_{k+1}c_k + s_{k+1})) + \frac{1}{2}(P_k(S_{k+1}c_k + s_{k+1}))' R_k(P_k(S_{k+1}c_k + s_{k+1}))$$

PROOF:

The proof is carried out by renaming some matrices and then showing that the relations for the optimal strategies $\gamma_k^*(x_k)$ of Corollary (2) can be rewritten in the way stated above.

Let us start by renaming the matrices Z_k , ζ_k and n_k .

$$(3.47) \quad Z_k \hat{=} S_k ; \zeta_k \hat{=} s_k ; n_k \hat{=} q_k$$

Considering (3.47), the feedback matrices given by (3.35) and (3.36) can be rewritten as

$$P_k^* = (R_k + B_k S_{k+1} B_k)^{-1} B_k' S_{k+1} A_k$$

$$\alpha_k^* = (R_k + B_k' S_{k+1} B_k)^{-1} B_k' (S_{k+1} c_k + s_{k+1})$$

Using (3.42) in the two equations above gives

$$(3.48) \quad P_k^* = P_k S_{k+1} A_k$$

$$(3.49) \quad \alpha_k^* = P_k(S_{k+1}c_k + s_{k+1})$$

Substituting P_k^* and α_k^* in (3.34) with the help of (3.48) and (3.49) yields

$$(3.41) \quad \gamma_k^*(x_k) = -P_k S_{k+1} A_k x_k - P_k(s_{k+1} + S_{k+1}c_k)$$

Next we prove that S_k and s_k fulfill (3.43) and (3.44) respectively.

Taking the renaming and equation (3.48) into consideration, (3.37) gives

$$(3.43) \quad S_k = (A_k - B_k P_k S_{k+1} A_k)' S_{k+1} (A_k - B_k P_k S_{k+1} A_k) \\ + (P_k S_{k+1} A_k)' R_k (P_k S_{k+1} A_k) + Q_k ; S_T = Q_T$$

Now we show the correctness of equation (3.44). To do so we rewrite (3.38) taking consideration of (3.47), (3.48) and (3.49)

$$(3.44) \quad s_k = (A_k - B_k P_k S_{k+1} A_k)' [s_{k+1} + S_{k+1}(c_k - B_k P_k (s_{k+1} + S_{k+1}c_k))] \\ + (P_k S_{k+1} A_k)' R_k P_k (s_{k+1} + S_{k+1}c_k) ; s_{T+1} = 0$$

Eventually the correctness of the rewritten value function $L(1, x_1)$ given by (3.45) has to be shown. Making use of (3.47) in (3.39) and (3.40) respectively leads to

$$(3.45) \quad L(1, x_1) = \frac{1}{2} x_1' S_1 x_1 + s_1' x_1 + q_1$$

$$(3.50) \quad q_k = q_{k+1} + \frac{1}{2} (c_k - B_k \alpha_k^*)' S_{k+1} (c_k - B_k \alpha_k^*) \\ + s_{k+1}' (c_k - B_k \alpha_k^*) + \frac{1}{2} \alpha_k^{*'} R_k \alpha_k^*$$

Finally substituting α_k^* in (3.50) with the help of (3.49) yields

$$\begin{aligned}
 (3.46) \quad q_k = q_{k+1} &+ \frac{1}{2}(c_k - B_k P_k(S_{k+1}c_k + s_{k+1}))' S_{k+1} (c_k - B_k P_k(S_{k+1}c_k + s_{k+1})) \\
 &+ s_{k+1}' (c_k - B_k P_k(S_{k+1}c_k + s_{k+1})) + \frac{1}{2}(P_k(S_{k+1}c_k + s_{k+1}))' R_k (P_k(S_{k+1}c_k + s_{k+1})) \quad \square
 \end{aligned}$$

3.2 Feedback Stackelberg Equilibrium Solutions

This section is devoted to the derivation of the feedback Stackelberg equilibrium solution with one leader and arbitrarily many followers for affine-quadratic games. First a general result is stated about the existence of a Stackelberg equilibrium solution with one leader and arbitrarily many followers in n -person discrete-time deterministic infinite dynamic games of prespecified fixed duration (cf. Def. (1)) with feedback information pattern. Then this result is applied to affine-quadratic games and finally the feedback Stackelberg equilibrium solutions with one leader and one follower for affine-quadratic and linear-quadratic games are deduced as special cases.

3.2.1 Optimality conditions

In this subsection a theorem is stated which gives necessary and sufficient conditions for the existence of a feedback Stackelberg equilibrium solution with one leader and arbitrarily many followers. Results about feedback Stackelberg equilibria in infinite dynamic games first appeared in discrete time in the works of Simaan and Cruz (1973) [13], [14].

Theorem 4 *For an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with feedback information pattern, the set of strategies $\{\gamma_k^{i*}(x_k); k \in K, i \in N\}$ provides a feedback Stackelberg equilibrium solution with **P1** as the leader and **P2** ... **Pn** as followers if, and only if, functions $V^i(k, \cdot) : R^n \rightarrow R, k \in K, i \in N$, exist such that the following recursive relations are*

satisfied:

$$(3.51) \quad V^1(k, x_{k-1}) = \min_{u_k^1 \in U_k^1, u_k^2 \in R_k^2(u_k^1), \dots, u_k^n \in R_k^n(u_k^1)} [g_k^1(f_{k-1}(x_{k-1}, u_k^1, u_k^2, \dots, u_k^n), u_k^1, u_k^2, \dots, u_k^n), u_k^1, u_k^2, \dots, u_k^n, x_{k-1}) + V^1(k+1, f_{k-1}(x_{k-1}, u_k^1, u_k^2, \dots, u_k^n))] = g_k^1[\tilde{f}_{k-1}^1(x_{k-1}, \gamma_k^{1*}(x_{k-1})), \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}), x_{k-1}] + V^1[k+1, \tilde{f}_{k-1}^1(x_{k-1}, \gamma_k^{1*}(x_{k-1}))]; V^1(T+1, x_T) = 0$$

$$V^i(k, x_{k-1}) = \min_{u_k^i \in U_k^i} [g_k^i(\tilde{f}_{k-1}^i(x_{k-1}, u_k^i), \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{i-1*}(x_{k-1}), u_k^i, \gamma_k^{i+1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}), x_{k-1}) + V^i(k+1, \tilde{f}_{k-1}^i(x_{k-1}, u_k^i))] = g_k^i[\tilde{f}_{k-1}^i(x_{k-1}, \gamma_k^{i*}(x_{k-1})), \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}), x_{k-1}] + V^i[k+1, \tilde{f}_{k-1}^i(x_{k-1}, \gamma_k^{i*}(x_{k-1}))]; V^i(T+1, x_T) = 0; i \in \{2 \dots n\}$$

where R_k^i ($i \in \{2 \dots n\}$) is a singleton set defined by

$$R_k^i(u_k^1) = \{r_k^i(u_k^1) \in \Gamma_k^i : g_k^i(f_k(x_{k-1}, u_k^1, r_k^2(u_k^1), \dots, r_k^n(u_k^1)), u_k^1, r_k^2(u_k^1), \dots, r_k^n(u_k^1), x_{k-1}) + V^i[k+1, f(x_{k-1}, u_k^1, r_k^2(u_k^1), \dots, r_k^n(u_k^1))] = \min_{u_k^i} g_k^i(f_k(x_{k-1}, u_k^1, r_k^2(u_k^1), \dots, r_k^{i-1}(u_k^1), u_k^i, r_k^{i+1}(u_k^1), \dots, r_k^n(u_k^1)), u_k^1, r_k^2(u_k^1), \dots, r_k^{i-1}(u_k^1), u_k^i, r_k^{i+1}(u_k^1), \dots, r_k^n(u_k^1), x_{k-1}) + V^i[k+1, f(x_{k-1}, u_k^1, r_k^2(u_k^1), \dots, r_k^{i-1}(u_k^1), u_k^i, r_k^{i+1}(u_k^1), \dots, r_k^n(u_k^1))]\}$$

$$R_k^i(\gamma_k^{1*}) = \gamma_k^{i*}$$

$$\tilde{f}_{k-1}^1(x_{k-1}, z) \hat{=} f_{k-1}(x_{k-1}, z, r_k^2(z), \dots, r_k^n(z))$$

$$\tilde{f}_{k-1}^i(x_{k-1}, z) \hat{=} f_{k-1}(x_{k-1}, \gamma_k^{1*}(x_{k-1}), \dots, \gamma_k^{i-1*}(x_{k-1}), z, \gamma_k^{i+1*}(x_{k-1}), \dots, \gamma_k^{n*}(x_{k-1}))$$

Every such equilibrium solution is strongly time consistent, and the corresponding Stackelberg equilibrium cost for P_i is $V^i(1, x_0)$.

PROOF:

Theorem (4) can be proven in the same way as Theorem (2), bearing in mind that the leader additionally accounts for the influence of his strategy on the followers' strategies when minimizing his cost functional. \square

Remark 6 For n -person affine-quadratic dynamic games (cf. Def. (4)) the assumption of a "unique follower response" (R_k^i ($i \in \{2 \dots n\}$) is a singleton set) is met if the followers' cost functions are strictly convex over R^{m_i} .

3.2.2 Results for affine-quadratic games with one leader and arbitrarily many followers

In the following, the results of Theorem (4) are applied to an affine-quadratic dynamic game with one leader and arbitrarily many followers. Theorem (5) is a generalization of Corollary 7.2 in Başar and Olsder (1999, pp. 374-375)[2]. On the one hand a more general state equation and more general cost functionals are considered and on the other hand the number of followers is extended from one to arbitrarily many.

Furthermore in Proposition (3) the equivalence of the equations in Theorem (5) with terminologically different equations is shown.

Theorem 5 *An n -person affine-quadratic dynamic game (cf. Def. (4)) admits a unique feedback Stackelberg equilibrium solution with one leader and arbitrarily many followers if*

1. $Q_k^i \geq 0$, $R_k^{ii} > 0$ and $R_k^{ij} \geq 0$ (defined for $k \in K$, $i, j \in N$, $j \neq i$)
2. (3.58), (3.59), (3.61), (3.63) and (3.64) admit unique optimal solutions P_k^{i*} , α_k^{i*} , \bar{r}_k^i , W_k^i and w_k^i (defined for $k \in K$, $i \in \{2, \dots, n\}$)

If these conditions are satisfied, the unique equilibrium strategies are given by (3.55) and the corresponding feedback Stackelberg equilibrium cost for each player is stated in (3.65).⁹

$$(3.52) \quad f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k$$

$$(3.53) \quad L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$$

⁹ For all equations belonging to this theorem and its proof, $i \in N$ and $k \in K$ if nothing different is stated.

$$(3.54) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) + \\ \frac{1}{2}(\tilde{x}_k^{j'} Q_k^i \tilde{x}_k^j + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{j'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j$$

$$(3.55) \quad \gamma_k^{i*}(x_{k-1}) = u_k^{i*} = -P_k^{i*} x_{k-1} - \alpha_k^{i*}$$

$$(3.56) \quad P_k^{1*} = [(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j)' Z_k^1 (B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j) + R_k^{11} + \sum_{j \in 2 \dots n} \bar{r}_k^{j'} R_k^{1j} \bar{r}_k^j]^{-1} \\ [(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j)' Z_k^1 (A_k + \sum_{j \in 2 \dots n} B_k^j W_k^j) + \sum_{j \in 2 \dots n} \bar{r}_k^{j'} R_k^{1j} W_k^j]$$

$$(3.57) \quad \alpha_k^{1*} = [(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j)' Z_k^1 (B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j) + R_k^{11} \\ + \sum_{j \in 2 \dots n} \bar{r}_k^{j'} R_k^{1j} \bar{r}_k^j]^{-1} [(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j)' Z_k^1 (s_k - \sum_{j \in 2 \dots n} B_k^j w_k^j) \\ + \sum_{j \in 2 \dots n} \bar{r}_k^{j'} R_k^{1j} w_k^j - R_k^{11} \tilde{u}_k^{11} \\ - \sum_{j \in 2 \dots n} \bar{r}_k^{j'} R_k^{1j} \tilde{u}_k^{1j} + (B_k^1 + \sum_{j \in 2 \dots n} B_k^j \bar{r}_k^j)' (\zeta_k^1 - Q_k^1 \tilde{x}_k^1)]$$

$$(3.58) \quad P_k^{i*} = (R_k^{ii} + B_k^{i'} Z_k^i B_k^i)^{-1} [B_k^{i'} Z_k^i (A_k - B_k^1 P_k^{1*} \\ - \sum_{j \in 2 \dots n, j \neq i} B_k^j P_k^{j*})] = -W_T^i + \bar{r}_T^i P_T^{1*}; i \in \{2 \dots n\}$$

$$(3.59) \quad \alpha_k^{i*} = (R_k^{ii} + B_k^{i'} Z_k^i B_k^i)^{-1} [B_k^{i'} (Z_k^i (s_k - B_k^1 \alpha_k^{1*} - \sum_{j \in 2 \dots n, j \neq i} B_k^j \alpha_k^{j*}) \\ + \zeta_k^i - Q_k^i \tilde{x}_k^i) - R_k^{ii} \tilde{u}_k^{ii}] = -w_k^i + \bar{r}_k^i \alpha_k^{1*}; i \in \{2 \dots n\}$$

$$(3.60) \quad Z_{k-1}^i = (A_k - \sum_{j \in N} B_k^j P_k^{j*})' Z_k^i (A_k - \sum_{j \in N} B_k^j P_k^{j*}) \\ + \sum_{j \in N} P_k^{j*'} R_k^{ij} P_k^{j*} + Q_{k-1}^i; \quad Z_T^i = Q_T^i$$

$$(3.61) \quad \bar{r}_k^i = -(R_k^{ii} + B_k^{i'} Z_k^i B_k^i)^{-1} [B_k^{i'} Z_k^i (B_k^1 + \sum_{j \in 2 \dots n, j \neq i} B_k^j \bar{r}_k^j)]; \quad i \in \{2 \dots n\}$$

$$(3.62) \quad \zeta_{k-1}^i = (A_k - \sum_{j \in N} B_k^j P_k^{j*})' [\zeta_k^i + Z_k^i (s_k - \sum_{j \in N} B_k^j \alpha_k^{j*}) - Q_k^i \tilde{x}_k^i] \\ + \sum_{j \in N} P_k^{j*'} R_k^{ij} (\alpha_k^{j*} + \tilde{u}_k^{ij}); \quad \zeta_T^i = 0$$

$$(3.63) \quad W_k^i = -(R_k^{ii} + B_k^{i'} Z_k^i B_k^i)^{-1} [B_k^{i'} Z_k^i (\sum_{j \in 2 \dots n, j \neq i} B_k^j W_k^j + A_k)]; \quad i \in \{2 \dots n\}$$

$$(3.64) \quad w_k^i = -(R_k^{ii} + B_k^{i'} Z_k^i B_k^i)^{-1} [B_k^{i'} (Z_k^i (\sum_{j \in 2 \dots n, j \neq i} B_k^j w_k^j + s_k) \\ + \zeta_k^i - Q_k^i \tilde{x}_k^i) - R_k^{ii} \tilde{u}_k^{ii}]; \quad i \in \{2 \dots n\}$$

$$(3.65) \quad V^i(1, x_0) = \frac{1}{2} x_0' Z_0^i x_0 + \zeta_0^{i'} x_0 + n_0^i$$

$$(3.66) \quad n_{k-1}^i = n_k^i + \frac{1}{2} (s_k - \sum_{j \in N} B_k^j \alpha_k^{j*})' Z_k^i (s_k - \sum_{j \in N} B_k^j \alpha_k^{j*}) \\ + \zeta_k^{i'} (s_k - \sum_{j \in N} B_k^j \alpha_k^{j*}) + \frac{1}{2} \sum_{j \in N} \alpha_k^{j*'} R_k^{ij} \alpha_k^{j*} - \tilde{x}_k^{i'} Q_k^i (s_k - \sum_{j \in N} B_k^j \alpha_k^{j*}) \\ + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \alpha_k^{j*} + \frac{1}{2} (\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}); \quad n_T^i = 0$$

PROOF:¹⁰

The proof is done using an induction argument to show that the strategy of the leader and the strategies of the followers $\gamma_k^*(x_{k-1})$, given by (3.55), minimize¹¹ the strictly convex functionals (3.67) and (3.68) at each stage of the game. But the minimization of (3.67) and (3.68) is exactly (3.51) applied to the specific state equation and cost functionals of the above game and therefore, considering Theorem (4), it follows that the $\gamma_k^*(x_{k-1})$ are the unique equilibrium strategies.

$$\begin{aligned}
 (3.67) \quad & \frac{1}{2}((A_k x_{k-1} + B_k^1 u_k^1 + \sum_{j \in \{2 \dots n\}} B_k^j r_k^j + s_k)' Q_k^1 (A_k x_{k-1} + B_k^1 u_k^1 \\
 & + \sum_{j \in \{2 \dots n\}} B_k^j r_k^j + s_k) + u_k^{1'} R_k^{11} u_k^1 + \sum_{j \in \{2 \dots n\}} r_k^{j'} R_k^{1j} r_k^j) + \frac{1}{2}(\tilde{x}_k^{1'} Q_k^1 \tilde{x}_k^1 \\
 & + \sum_{j \in \{2 \dots n\}} \tilde{u}_k^{1j'} R_k^{1j} \tilde{u}_k^{1j}) - \tilde{x}_k^{1'} Q_k^1 (A_k x_{k-1} + B_k^1 u_k^1 + \sum_{j \in \{2 \dots n\}} B_k^j r_k^j + s_k) - \tilde{u}_k^{11'} R_k^{11} u_k^1 \\
 & - \sum_{j \in \{2 \dots n\}} \tilde{u}_k^{1j'} R_k^{1j} r_k^j + V^1(k+1, \tilde{f}_{k-1}^1(x_{k-1}, u_k^1)) ; V^1(T+1, x_T) = 0
 \end{aligned}$$

$$\begin{aligned}
 (3.68) \quad & \frac{1}{2}((A_k x_{k-1} + B_k^i u_k^i + \sum_{j \in N, j \neq i} B_k^j u_k^{j*} + s_k)' Q_k^i (A_k x_{k-1} + B_k^i u_k^i \\
 & + \sum_{j \in N, j \neq i} B_k^j u_k^{j*} + s_k) + u_k^{i'} R_k^{ii} u_k^i + \sum_{j \in N, j \neq i} u_k^{j*'} R_k^{ij} u_k^{j*}) + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i \\
 & + \sum_{j \in N, j \neq i} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i (A_k x_{k-1} + B_k^i u_k^i + \sum_{j \in N, j \neq i} B_k^j u_k^{j*} + s_k) - \tilde{u}_k^{ii'} R_k^{ii} u_k^i \\
 & - \sum_{j \in N, j \neq i} \tilde{u}_k^{ij'} R_k^{ij} u_k^{j*} + V^i(k+1, \tilde{f}_{k-1}^i(x_{k-1}, u_k^i)) ; V^i(T+1, x_T) = 0 ; i \in \{2 \dots n\}
 \end{aligned}$$

Furthermore the optimal reactions r_k^j of the followers to an arbitrary strategy u_k^1 of the leader at stage k , subject to the assumption that all strategies for all players from

¹⁰ The basis and the last part of the inductive step (after finding the optimal strategies for the leader and the followers) are proven in the same way as in Theorem (3), because in these parts the distinction between leader and followers is not essential.

¹¹ Taking into consideration the structural advantage of the leader, of course.

stage $k+1$ to stage T are optimal, are derived by minimization over u_l^i of the below equations

$$\begin{aligned}
 (3.69) \quad & \frac{1}{2}((A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l)' Q_l^i (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i \\
 & + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) + u_l^{i'1} R_l^{i1} u_l^1 + u_l^{i'j} R_l^{ij} u_l^j + \sum_{j \in \{2 \dots n\}, j \neq i} r_l^{j'} R_l^{ij} r_l^j) \\
 & + \frac{1}{2}(\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i \\
 & + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) - \tilde{u}_l^{i'1} R_l^{i1} u_l^1 - \tilde{u}_l^{i'j} R_l^{ij} u_l^j - \sum_{j \in \{2 \dots n\}, j \neq i} \tilde{u}_l^{ij'} R_l^{ij} r_l^j \\
 & + V^i(k+1, f_{k-1}^i(x_{k-1}, u_k^1, r_k^2, \dots, r_k^{i-1}, u_k^i, r_k^{i+1}, \dots, r_k^n)) ; i \in \{2 \dots n\}
 \end{aligned}$$

The induction argument runs from $T+1$ to 1 and proves that the value function for player i at stage k can be written as stated below in (3.70).

$$(3.70) \quad \frac{1}{2}x_{k-1}'(Z_{k-1}^i - Q_{k-1}^i)x_{k-1} + \zeta_{k-1}^{i'}x_{k-1} + n_{k-1}^i$$

Basis:

The induction starts at $k = T$. First we make use of the general optimality conditions for V_k^i at stage $T+1$.

$$(3.71) \quad V^i(T+1, x_T) = 0$$

Now we show that $(3.70)_{k=T+1}$ fulfills (3.71).

$$(3.70)_{k=T+1} \quad \frac{1}{2}x_T'(Z_T^i - Q_T^i)x_T + \zeta_T^{i'}x_T + n_T^i$$

Making use of $(3.60)_{k=T+1}$, $(3.62)_{k=T+1}$ and $(3.66)_{k=T+1}$ yields

$$\frac{1}{2}x_T'(Q_T^i - Q_T^i)x_T + 0'x_T + 0 = 0 = V^i(T+1, x_T)$$

Inductive step:

As an induction hypothesis, the equations $(3.70)_{k=l+1}$ are assumed to be equal to $V^i(l+1, x_l)$ respectively. Now we have to prove that the relation also holds at stage l . In other words: we have to show $(3.70)_{k=l} = V^i(l, x_{l-1})$.

$$(3.70)_{k=l+1} \quad \frac{1}{2}x_l'(Z_l^i - Q_l^i)x_l + \zeta_l^{i'}x_l + n_{l-1}^i$$

The inductive step is done by first showing what the optimal response of the followers to an arbitrary strategy of the leader looks like. In other words: first we deduce the optimal reactions r_l^j ($j \in \{2, \dots, n\}$) of the followers. Then we derive the optimal strategy of the leader by minimizing $(3.67)_{k=l}$ over u_l^1 considering the optimal reactions of the followers, and finally we derive the optimal strategies of the followers as the optimal reactions of the followers to the optimal strategy of the leader.

Using $(3.52)_{k=l}$ in $(3.70)_{k=l+1}$, bearing in mind that the optimal control vector of the leader remains to be deduced in the further induction argument, gives

$$(3.72) \quad \frac{1}{2}(A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l)'(Z_l^1 - Q_l^1)(A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) + \zeta_l^{1'}(A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) + n_l^1$$

for the leader and

$$(3.73) \quad \frac{1}{2}(A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l)' (Z_l^i - Q_l^i) \\ (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) + \zeta_l^{i'} (A_l x_{l-1} + B_l^1 u_l^1 \\ + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^{j*} + s_l) + n_l^i ; i \in \{2 \dots n\}$$

for the followers.

We start with the derivation of the optimal reactions of the followers r_l^i ($i \in \{2, \dots, n\}$) to an arbitrary strategy u_l^1 of the leader. To do this we substitute $V^i(l+1, \tilde{f}_{k-1}^i(x_{k-1}, u_k^1, r_k^2, \dots, r_k^{i-1}, u_k^i, r_k^{i+1}, \dots, r_k^n))$ ($i \in \{2, \dots, n\}$) in (3.69)_{k=l} with the help of the induction hypothesis given by (3.73):

$$(3.74) \quad \frac{1}{2}((A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in N, j \neq i} B_l^j r_l^j + s_l)' Q_l^i (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i \\ + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) + u_l^{i'} R_l^{i1} u_l^1 + u_l^{i'} R_l^{ii} u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} r_l^{j'} R_l^{ij} r_l^j) + \frac{1}{2}(\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i \\ + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) - \\ \tilde{u}_l^{i1'} R_l^{i1} u_l^1 - \tilde{u}_l^{ii'} R_l^{ii} u_l^i - \sum_{j \in \{2 \dots n\}, j \neq i} \tilde{u}_l^{ij'} R_l^{ij} r_l^j + \frac{1}{2}(A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \\ \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l)' (Z_l^i - Q_l^i) (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) \\ + \zeta_l^{i'} (A_l x_{l-1} + B_l^1 u_l^1 + B_l^i u_l^i + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j + s_l) + n_l^i ; i \in \{2 \dots n\}$$

(3.74) is strictly convex in u_l^i . This can be seen by applying Corollary (1) to (3.75). Therefore there has to be a unique optimal response of follower i ($i \in \{2, \dots, n\}$) to

an arbitrary strategy u_l^1 of the leader.

$$(3.75) \quad \frac{\partial^2}{\partial u_l^2} (3.74) \quad R_l^{ii} + B_l^{i'} Z_l^i B_l^i ; i \in \{2 \dots n\}$$

This unique optimal response can be found by using the first-order necessary and sufficient (because of strict convexity of (3.74)) conditions for minimization

$$\begin{aligned} \frac{\partial}{\partial u_l^i} (3.74) = 0 \Rightarrow & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i) r_l^i - B_l^{i'} Z_l^i \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j = \\ & B_l^{i'} [Z_l^i (A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i] - R_l^{ii} \tilde{u}_l^{ii} + B_l^{i'} Z_l^i B_l^1 u_l^1 ; i \in \{2 \dots n\} \end{aligned}$$

Making r_l^i explicit leads to

$$\begin{aligned} (3.76) \quad r_l^i = & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j r_l^j \\ & + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\} \end{aligned}$$

As the right hand side of the above equation is affine in x_{l-1} and u_l^1 , the left hand side also has to be affine in x_{l-1} and u_l^1 . Therefore the substitution

$$(3.77) \quad r_l^i = W_l^i x_{l-1} + \bar{r}_l^i u_l^1 + w_l^i ; i \in \{2 \dots n\}$$

is allowed and leads to

$$\begin{aligned} (3.78) \quad W_l^i x_{l-1} + \bar{r}_l^i u_l^1 + w_l^i = & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (B_l^1 u_l^1 \\ & + \sum_{j \in \{2 \dots n\}, j \neq i} B_l^j (W_l^j x_{l-1} + \bar{r}_l^j u_l^1 + w_l^j) + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\} \end{aligned}$$

Differentiating (3.77) gives

$$\frac{\partial}{\partial u_l^1}(3.77) \quad \frac{\partial}{\partial u_l^1} r_l^i = \bar{r}_l^i ; i \in \{2 \dots n\}$$

Differentiating (3.78) and using the above equation yields

$$(3.79) \quad \frac{\partial}{\partial u_l^1}(3.78) \quad \bar{r}_l^i = -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} \\ [B_l^{i'} Z_l^i (B_l^1 + \sum_{j \in 2 \dots n, j \neq i} B_l^j \bar{r}_l^j)] ; i \in \{2 \dots n\}$$

Rearranging the terms in (3.77) gives

$$W_l^i x_{l-1} + w_l^i = r_l^i - \bar{r}_l^i u_l^1 ; i \in \{2 \dots n\}$$

Now we substitute r_l^i and \bar{r}_l^i with the help of (3.76) and (3.79)

$$W_l^i x_{l-1} + w_l^i = -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} \\ [B_l^{i'} (Z_l^i (B_l^1 u_l^1 + \sum_{j \in 2 \dots n, j \neq i} B_l^j r_l^j + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] \\ + (R_l^{ii} + B_l^{i'} Q_l^i B_l^i)^{-1} [-B_l^{i'} Z_l^i (B_l^1 + \sum_{j \in 2 \dots n, j \neq i} B_l^j \bar{r}_l^j)] u_l^1 ; i \in \{2 \dots n\}$$

Making use of (3.77) leads to

$$(3.80) \quad W_l^i x_{l-1} + w_l^i = -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (\sum_{j \in 2 \dots n, j \neq i} B_l^j \\ (W_l^j x_{l-1} + w_l^j) + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\}$$

Comparing the coefficients it follows that

$$(3.80)_{x_{l-1}} = (3.63)_{k=l} \quad W_l^i = (R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} Z_l^i (\sum_{j \in 2 \dots n, j \neq i} B_l^j W_l^j + A_l)] ; i \in \{2 \dots n\}$$

$$(3.80)_{const.} = (3.64)_{k=l} \quad w_l^i = -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (\sum_{j \in 2 \dots n, j \neq i} B_l^j w_l^j + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\}$$

After deducing the unique optimal response of follower i ($i \in \{2, \dots, n\}$) to an arbitrary strategy u_l^1 of the leader, we can substitute $V^1(k+1, \tilde{f}_{k-1}^1(x_{k-1}, u_k^1))$ in $(3.67)_{k=l}$ ($i \in \{2, \dots, n\}$) with the help of the induction hypothesis for the leader, given by (3.72).

$$\begin{aligned} (3.67)_{k=l} \quad & \frac{1}{2} ((A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l)' Q_l^1 (A_l x_{l-1} + B_l^1 u_l^1 \\ & + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) + u_l^{1'} R_l^{11} u_l^1 + \sum_{j \in \{2 \dots n\}} r_l^{j'} R_l^{1j} r_l^j) + \frac{1}{2} (\tilde{x}_l^{1'} Q_l^1 \tilde{x}_l^1 \\ & + \sum_{j \in \{2 \dots n\}} \tilde{u}_l^{1j'} R_l^{1j} \tilde{u}_l^{1j}) - \tilde{x}_l^{1'} Q_l^1 (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) - \tilde{u}_l^{11'} R_l^{11} u_l^1 \\ & - \sum_{j \in \{2 \dots n\}} \tilde{u}_l^{1j'} R_l^{1j} r_l^j + V^1(l+1, \tilde{f}_{l-1}^1(x_{l-1}, u_l^1)) \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2}((A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l)' Q_l^1 (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) \\
 & \quad + u_l^{1'} R_l^{11} u_l^1 + \sum_{j \in \{2 \dots n\}} r_l^{j'} R_l^{1j} r_l^j) + \frac{1}{2}(\tilde{x}_l^{1'} Q_l^1 \tilde{x}_l^1 + \sum_{j \in \{2 \dots n\}} \tilde{u}_l^{1j'} R_l^{1j} \tilde{u}_l^{1j}) \\
 & \quad - \tilde{x}_l^{1'} Q_l^1 (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) - \tilde{u}_l^{11'} R_l^{11} u_l^1 - \sum_{j \in \{2 \dots n\}} \tilde{u}_l^{1j'} R_l^{1j} r_l^j \\
 & + \frac{1}{2}(A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l)' (Z_l^1 - Q_l^1) (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) \\
 & \quad + \zeta_l^{1'} (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in \{2 \dots n\}} B_l^j r_l^j + s_l) + n_l^1
 \end{aligned}$$

Next we substitute r_k^i using (3.77)

$$\begin{aligned}
 (3.81) \quad & \frac{1}{2}((A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in 2 \dots n} B_l^j (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + s_l)' \\
 & Z_l^1 (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in 2 \dots n} B_l^j (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + s_l) + u_l^{1'} R_l^{11} u_l^1 + \\
 & \sum_{j \in 2 \dots n} r_l^{j'} R_l^{1j} (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1)) + \frac{1}{2}(\tilde{x}_l^{1'} Q_l^1 \tilde{x}_l^1 + \sum_{j \in N} \tilde{u}_l^{1j'} R_l^{1j} \tilde{u}_l^{1j}) - \tilde{x}_l^{1'} Q_l^1 (A_l x_{l-1} + B_l^1 u_l^1 + \\
 & \sum_{j \in 2 \dots n} B_l^j (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + s_l) - \tilde{u}_l^{11'} R_l^{11} u_l^1 - \sum_{j \in 2 \dots n} \tilde{u}_l^{1j'} R_l^{1j} (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + \\
 & \frac{1}{2}(A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in 2 \dots n} B_l^j (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + s_l)' (Z_l^1 - Q_l^1) \\
 & (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in 2 \dots n} B_l^j (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + s_l) + \\
 & \zeta_l^{1'} (A_l x_{l-1} + B_l^1 u_l^1 + \sum_{j \in 2 \dots n} B_l^j (W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^1) + s_l) + n_l^1
 \end{aligned}$$

(3.67) $_{k=l}$ is strictly convex in u_l^1 . This can be seen by applying Corollary (1) to (3.82). Therefore there has to be a unique optimal strategy of the leader at stage

l .

$$(3.82) \quad \frac{\partial^2}{\partial u_l^2} (3.81) \quad R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \bar{r}_l^j + (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)$$

This unique optimal strategy for the leader can be found by using the first-order necessary and sufficient (because of strict convexity of $(3.67)_{k=l}$) conditions for minimization

$$\begin{aligned} \frac{\partial}{\partial u_l^1} (3.81) = 0 \Rightarrow & (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (A_l x_{l-1} + B_l^1 u_l^{1*} + \sum_{j \in 2 \dots n} B_l^j \\ & (W_l^j x_{l-1} + w_l^j + \bar{r}_l^j u_l^{1*}) + s_l) + R_l^{11} u_l^{1*} + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} (W_l^j x_{l-1} + w_l^j + \\ & \bar{r}_l^j u_l^{1*}) - (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Q_l^1 \tilde{x}_l^1 - R_l^{11} \tilde{u}_l^{11} - \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \tilde{u}_l^{1j} + (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' \zeta_l^1 = 0 \end{aligned}$$

Rearranging the terms in the above equation yields

$$\begin{aligned} & - (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j) u_l^{1*} - (R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \bar{r}_l^j) u_l^{1*} \\ = & (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (A_l x_{l-1} + \sum_{j \in 2 \dots n} B_l^j (W_l^j x_{l-1} + w_l^j) + s_l) + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} (W_l^j x_{l-1} + w_l^j) \\ & - (B_l^1 + \bar{r}_l^j)' Q_l^1 \tilde{x}_l^1 - R_l^{11} \tilde{u}_l^{11} - \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \tilde{u}_l^{1j} + (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' \zeta_l^1 \end{aligned}$$

The structure of the above equation justifies the following substitution

$$(3.83) \quad (3.55)_{k=l}^{i=1} u_l^{1*} = -P_l^{1*} x_{l-1} - \alpha_l^{1*}$$

$$\begin{aligned}
 (3.84) \quad & (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j) (P_k^{1*} x_{k-1} + \alpha_k^{1*}) \\
 & + (R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \bar{r}_l^j) (P_k^{1*} x_{k-1} + \alpha_k^{1*}) = (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' \\
 & Z_l^1 (A_l x_{l-1} + \sum_{j \in 2 \dots n} B_l^j (W_l^j x_{l-1} + w_l^j) + s_l) + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} (W_l^j x_{l-1} + w_l^j) \\
 & - (B_l^1 + \bar{r}_l^j)' Q_l^1 \tilde{x}_l^1 - R_l^{11} \tilde{u}_l^{11} - \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} \tilde{u}_l^{1j} + (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' \zeta_l^1
 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}
 (3.84)_{x_l} \quad & [(B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j) + R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \bar{r}_l^j] P_k^{1*} = \\
 & (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (A_l + \sum_{j \in 2 \dots n} B_l^j W_l^j) + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} W_l^j
 \end{aligned}$$

$$\begin{aligned}
 (3.84)_{const.} \quad & [(B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j) + R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^j R_l^{1j} \bar{r}_l^j] \alpha_k^{1*} = \\
 & (B_l^1 + \bar{r}_l^j)' Z_l^1 (\sum_{j \in 2 \dots n} B_l^j w_l^j + s_l) + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} w_l^j \\
 & - (B_l^1 + \bar{r}_l^j)' Q_l^1 \tilde{x}_l^1 - R_l^{11} \tilde{u}_l^{11} - \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} \tilde{u}_l^{1j} + (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' \zeta_l^1
 \end{aligned}$$

Making P_l^{1*} and α_l^{1*} explicit finally leads to

$$\begin{aligned}
 (3.56)_{k=l} \quad & P_l^{1*} = [(B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j) + R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} \bar{r}_l^j]^{-1} \\
 & [(B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (A_l - \sum_{j \in 2 \dots n} B_l^j W_l^j) + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} W_l^j]
 \end{aligned}$$

$$\begin{aligned}
 (3.57)_{k=l} \quad \alpha_l^{1*} = & [(B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j) + R_l^{11} + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} \bar{r}_l^j]^{-1} \\
 & [(B_l^1 + \sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' Z_l^1 (\sum_{j \in 2 \dots n} B_l^j w_l^j + s_l) + \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} w_l^j \\
 & - R_l^{11} \tilde{u}_l^{11} - \sum_{j \in 2 \dots n} \bar{r}_l^{j'} R_l^{1j} \tilde{u}_l^{1j} + (B_l^1 + (\sum_{j \in 2 \dots n} B_l^j \bar{r}_l^j)' (\zeta_l^1 - Q_l^1 \tilde{x}_l^1))]
 \end{aligned}$$

By making use of the unique optimal strategy u_l^{1*} of the leader in the optimal response functions of the followers given by (3.76) and, in a different presentation, by (3.77) we get the optimal strategies of the followers

$$\begin{aligned}
 r_l^{i*} = u_l^{i*} = & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (B_l^1 u_l^{1*} + \sum_{j \in 2 \dots n, j \neq i} B_l^j u_l^{j*} \\
 & + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\}
 \end{aligned}$$

$$r_l^{i*} = u_l^{i*} = W_l^i x_{l-1} + w_l^i + \bar{r}_l^i u_l^{1*} ; i \in \{2 \dots n\}$$

Making use of (3.83) in the above two systems of equations yields

$$\begin{aligned}
 u_l^{i*} = & -(R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (B_l^1 (P_l^{1*} x_{l-1} + \alpha_l^{1*}) + \sum_{j \in 2 \dots n, j \neq i} B_l^j u_l^{j*} \\
 & + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\}
 \end{aligned}$$

$$u_l^{i*} = W_l^i + w_l^i - \bar{r}_l^i (P_l^{1*} x_{l-1} + \alpha_l^{1*}) ; i \in \{2 \dots n\}$$

The structure of the above two systems of equations justifies the following substitution

$$(3.85) \quad (3.55)_{k=l} \quad u_l^{i*} = -P_k^{i*} x_{k-1} - \alpha_k^{i*} ; i \in \{2 \dots n\}$$

$$(3.86) \quad P_k^{i*} x_{k-1} + \alpha_k^{i*} = (R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i (P_l^{1*} x_{l-1} + \alpha_l^{1*}) + \sum_{j \in 2 \dots n, j \neq i} B_l^j (-P_k^{i*} x_{k-1} - \alpha_k^{i*}) + A_l x_{l-1} + s_l) + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] ; i \in \{2 \dots n\}$$

$$(3.87) \quad P_k^{i*} x_{k-1} + \alpha_k^{i*} = -W_l^i - w_l^i + \bar{r}_l^i (P_l^{1*} x_{l-1} + \alpha_l^{1*}) ; i \in \{2 \dots n\}$$

By comparing coefficients it follows that

$$(3.86)_{x_l} = (3.87)_{x_l} = (3.58)_{k=l} \quad P_l^{i*} = (R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} Z_l^i (A_l - B_l^1 P_l^{1*} - \sum_{j \in 2 \dots n, j \neq i} B_l^j P_l^{j*})] = -W_l^i + \bar{r}_l^i P_l^{1*} ; i \in \{2 \dots n\}$$

$$(3.86)_{const.} = (3.87)_{const.} = (3.59)_{k=l} \quad \alpha_l^{i*} = (R_l^{ii} + B_l^{i'} Z_l^i B_l^i)^{-1} [B_l^{i'} (Z_l^i s_l - B_l^1 \alpha_l^{1*} - \sum_{j \in 2 \dots n, j \neq i} B_l^j \alpha_l^{j*} + \zeta_l^i - Q_l^i \tilde{x}_l^i) - R_l^{ii} \tilde{u}_l^{ii}] = -w_l^i + \bar{r}_l^i \alpha_l^{1*} ; i \in \{2 \dots n\}$$

Now, after finding the optimal strategies for the players, we are able to rewrite (3.72) and (3.73) as

$$\begin{aligned} V^i(l, x_{l-1}) = & \frac{1}{2} [(A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l)' Z_l^i (A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l) \\ & + \sum_{j \in N} u_l^{j*'} R_l^{ij} u_l^{j*}] + \frac{1}{2} (\tilde{x}_l^{i'} Q_l^i \tilde{x}_l^i + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l^{i'} Q_l^i (A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l) \\ & - \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} u_l^{j*} + \zeta_l^{i'} (A_l x_{l-1} + \sum_{j \in N} B_l^j u_l^{j*} + s_l) + n_l \end{aligned}$$

Making use of (3.85) and (3.83) respectively yields

$$\begin{aligned}
 V^i(l, x_{l-1}) = & \frac{1}{2}[(A_l x_{l-1} + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l)' Z_l^i (A_l x_{l-1} \\
 & + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l) + \sum_{j \in N} (-P_l^{j*} x_{l-1} - \alpha_l^{j*})' R_l^{ij} (-P_l^{j*} x_{l-1} - \alpha_l^{j*})] \\
 & + \frac{1}{2}(\tilde{x}_l' Q_l^i \tilde{x}_l + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) - \tilde{x}_l' Q_l^i (A_l x_{l-1} + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l) \\
 & - \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + \zeta_l^{i'} (A_l x_{l-1} + \sum_{j \in N} B_l^j (-P_l^{j*} x_{l-1} - \alpha_l^{j*}) + s_l) + n_l
 \end{aligned}$$

Rewriting the above equation to the power of x_{l-1} gives

$$\begin{aligned}
 V^i(l, x_{l-1}) = & \frac{1}{2} x_{l-1}' [(A_l - \sum_{j \in N} B_l^j P_l^{j*})' Q_l^i (A_l - \sum_{j \in N} B_l^j P_l^{j*}) + \sum_{j \in N} P_l^{j*'} R_l^{ij} P_l^{j*}] x_{l-1} \\
 & + [(A_l - \sum_{j \in N} B_l^j P_l^{j*})' [\zeta_l^i Z_l^i (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) - Q_l^i \tilde{x}_l] + \sum_{j \in N} P_l^{j*'} R_l^{ij} (\alpha_l^{j*} + \tilde{u}_l^{ij})]' x_{l-1} \\
 & + \frac{1}{2} (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*})' Z_l^i (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) + \frac{1}{2} \sum_{j \in N} \alpha_l^{j*'} R_l^{ij} \alpha_l^{j*} \\
 & - \tilde{x}_l' Q_l^i (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \alpha_l^{j*} + \frac{1}{2} (\tilde{x}_l' Q_l^i \tilde{x}_l + \sum_{j \in N} \tilde{u}_l^{ij'} R_l^{ij} \tilde{u}_l^{ij}) \\
 & + \zeta_l^{i'} (s_l - \sum_{j \in N} B_l^j \alpha_l^{j*}) + n_l
 \end{aligned}$$

To finish off the inductive step and consequently the induction argument, we use the recursive equations (3.60)_{k=l}, (3.62)_{k=l} and (3.66)_{k=l} in the above equation. This leads to

$$(3.70)_{k=l} \quad V^i(l, x_{l-1}) = \frac{1}{2} x_{l-1}' (Z_{l-1}^i - Q_{l-1}^i) x_{l-1} + \zeta_{l-1}^{i'} x_{l-1} + n_{l-1}^i$$

The expression for the total costs of the game for player i given by (3.10) is equal to the function the induction argument was based on at stage 1. In other words, (3.10) is equal to (3.70)_{k=1}. \square

Remark 7 The proof of Theorem (5) is a formalization and generalization of the heuristic argumentation presented in Başar and Olsder (1999, pp. 274-275)[2].

Remark 8 Z_k^i given by (3.60) is positive definite for all $k \in \{0, \dots, T\}$. This can be proven by a straightforward induction argument, starting at stage T and using Corollary (1) in the inductive step.

Remark 9 To solve the Stackelberg game algorithmically, the following order of application of the equations of Theorem (5) is advisable ($i \in N$, $j \in \{2, \dots, n\}$):

1. For k running backward from T to 1

a) Z_k^i , ζ_k^i and n_k^i

b) \bar{r}_k^j , W_k^j and w_k^j

c) P_k^{1*} , α_k^{1*}

d) P_k^{j*} , α_k^{j*}

2. Z_0^i , ζ_0^i and n_0^i

3. $V^i(1, x_0)$

4. For k running forward from 1 to T

$\gamma_k^{i*}(x_{k-1})$

Proposition 3¹² The systems of equations defining the unique equilibrium strategies $\gamma_k^{i*}(x_{k-1})$ in Theorem (5) can also be written in the following way:¹³

$$(3.88) \quad \gamma_k^{i*}(x_{k-1}) = G_k^{i*} x_{k-1} + g_k^{i*}$$

$$(3.89) \quad G_k^{1*} = -[\Lambda]^{-1} [\bar{B}_k' H_k^1 A_k + \sum_{j \in 2 \dots n} \bar{D}_k^j W_k^j]$$

¹² In this proposition we rewrite the equilibrium equations in a notation that was used at our department in the past to enable comparison.

¹³ For all equations belonging to this proposition and its proof, $k \in K$ and $i \in N$ if nothing different is stated.

$$(3.90) \quad g_k^{1*} = -[\bar{\Lambda}_k]^{-1}[\bar{v}_k^1 + \bar{v}_k + \sum_{j \in 2 \dots n} \bar{D}_k^j w_k^j]$$

$$(3.91) \quad G_k^{i*} = W_k^i + \Psi_k^i G_k^{1*}; i \in \{2 \dots n\}$$

$$(3.92) \quad g_k^{i*} = w_k^i + \Psi_k^i g_k^{1*}; i \in \{2 \dots n\}$$

$$(3.93) \quad \bar{\Lambda}_k = \bar{B}_k' H_k^1 \bar{B}_k + R_k^{11} + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} \Psi_k^j$$

$$(3.94) \quad \bar{B}_k = B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j$$

$$(3.95) \quad \Psi_k^i = -(D_k^i)^{-1} [B_k^{i'} Z_k^i (B_k^1 + \sum_{j \in 2 \dots n, j \neq i} B_k^j \Psi_k^j)]; i \in \{2 \dots n\}$$

$$(3.96) \quad H_{k-1}^i = K_k' Z_k^i K_k + \sum_{j \in N} G_k^{j'} R_k^{ij} G_k^j + Q_{k-1}^i; H_T^i = Q_T^i$$

$$(3.97) \quad h_{k-1}^i = Q_{k-1}^i \tilde{x}_{k-1}^i - K_k' [H_k^i k_k - h_k^i] + \sum_{j \in N} G_k^{j'} R_k^{ij} (\tilde{u}_k^{ij} - g_k^j); h_T^i = Q_T^i \tilde{x}_T^i$$

$$(3.98) \quad K_k = A_k + \sum_{j \in N} B_k^j G_k^j$$

$$(3.99) \quad k_k = s_k + \sum_{j \in N} B_k^j g_k^j$$

$$(3.100) \quad D_k^i = R_k^{ii} + B_k^{i'} H_k^i B_k^i$$

$$(3.101) \quad \bar{D}_k = \bar{B}_k' H_k^1 \bar{B}_k + \Psi_k^{i'} R_k^{1i}$$

$$(3.102) \quad W_k^i = -(D_k^i)^{-1} [B_k^{i'} H_k^i (\sum_{j \in 2 \dots n, j \neq i} B_k^j W_k^j + A_k)]; i \in \{2 \dots n\}$$

$$(3.103) \quad w_k^i = -(D_k^i)^{-1} [B_k^{i'} H_k^i \sum_{j \in 2 \dots n, j \neq i} B_k^j w_k^j + v_k^i]; i \in \{2 \dots n\}$$

$$(3.104) \quad v_k^i = B_k^{i'} (H_k^i s_k - h_k^i) - R_k^{ii} \tilde{u}_k^{ii}; i \in N$$

$$(3.105) \quad \bar{v}_k = \sum_{j \in 2 \dots n} \Psi_k^{j'} (B_k^{j'} H_k^1 s_k - R_k^{1j} \tilde{u}_k^{1j} - B_k^{j'} h_k^1)$$

PROOF:

The proof is carried out by renaming some matrices and then showing that the relations for the equilibrium strategies $\gamma_k^{i*}(x_{k-1})$ of Theorem (5) can be rewritten in the way stated above.

Let us start by renaming the feedback matrices P_k^{i*} and α_k^{i*} , the reaction coefficients \bar{r}_k^j ($j \in \{2, \dots, n\}$) and the matrices Z_k^i and ζ_k^i .

$$(3.106) \quad P_k^i \triangleq -G_k^i; \alpha_k^i \triangleq -g_k^i; \bar{r}_k^j \triangleq \Psi_k^j; Z_k^i \triangleq H_k^i; \\ \zeta_k^i \triangleq -h_k^i + Q_k^i \tilde{x}_k^i; j \in \{2, \dots, n\}$$

Next we prove that Z_k^i and ζ_k^i fulfill (3.96) and (3.97) respectively.

Taking consideration of the renaming (3.60) gives

$$H_{k-1}^i = (A_k + \sum_{j \in N} B_k^i G_k^j)' H_k^i (A_k + \sum_{j \in N} B_k^j G_k^j) + \sum_{j \in N} G_k^{j'} R_k^{ij} G_k^j + Q_{k-1}^i$$

Making use of (3.98) yields

$$(3.96) \quad H_{k-1}^i = K_k' H_k^i K_k + \sum_{j \in N} G_k^{j'} R_k^{ij} G_k^j + Q_{k-1}^i$$

Now we demonstrate the correctness of equation (3.97). To do so we start at stage T and use (3.62) and (3.106) to get

$$0 = \zeta_T^i = -h_T^i + Q_T^i \tilde{x}_T^i$$

$$(3.97)_{k=T} \quad h_T^i = Q_T^i \tilde{x}_T^i$$

For the general stage k rewrite (3.62) taking consideration of (3.106)

$$-h_{k-1}^i + Q_{k-1}^i \tilde{x}_{k-1}^i = (A_k + \sum_{j \in N} B_k^j G_k^j)' [-h_k^i + H_k^i (s_k + \sum_{j \in N} B_k^j g_k^j)] - \sum_{j \in N} G_k^{j'} R_k^{ij} (\tilde{u}_k^{ij} - g_k^j)$$

Using (3.98) and (3.99) and making h_{k-1}^i explicit yields

$$(3.97) \quad h_{k-1}^i = Q_{k-1}^i \tilde{x}_{k-1}^i - K_k' [H_k^i k_k - h_k^i] + \sum_{j \in N} G_k^{j'} R_k^{ij} (\tilde{u}_k^{ij} - g_k^j)$$

Furthermore the relations for W_k^i , w_k^i and Ψ_k^i ($i \in \{2, \dots, n\}$), given by (3.102), (3.103) and (3.95), have to be deduced.

Making use of (3.106) and (3.100) in (3.63) leads to

$$W_k^i = -(R_k^{ii} + B_k^{i'} H_k^i B_k^i)^{-1} [B_k^{i'} H_k^i (\sum_{j \in \{2, \dots, n\}, j \neq i} B_k^j W_k^j + A_k)]; \quad i \in \{2, \dots, n\}$$

$$(3.102) \quad W_k^i = -(D_k^i)^{-1} [B_k^{i'} H_k^i (\sum_{j \in 2 \dots n, j \neq i} B_k^j W_k^j + A_k)] ; i \in \{2 \dots n\}$$

Using (3.106) and (3.100) in (3.64) yields

$$w_k^i = -(R_k^{ii} + B_k^{i'} H_k^i B_k^i)^{-1} [B_k^{i'} (H_k^i (\sum_{j \in 2 \dots n, j \neq i} B_k^j w_k^j + s_k) + h_k^i) - R_k^{ii} \tilde{u}_k^{ii}] ; i \in \{2 \dots n\}$$

$$(3.103) \quad w_k^i = -(D_k^i)^{-1} [B_k^{i'} H_k^i (\sum_{j \in 2 \dots n, j \neq i} B_k^j w_k^j + v_k^i)] ; i \in \{2 \dots n\}$$

Considering (3.106) and (3.100), (3.61) gives

$$\Psi_k^i = -(R_k^{ii} + B_k^{i'} H_k^i B_k^i)^{-1} [B_k^{i'} H_k^i (B_k^1 + \sum_{j \in 2 \dots n, j \neq i} B_k^j \Psi_k^j)] ; i \in \{2 \dots n\}$$

$$(3.95) \quad \Psi_k^i = -(D_k^i)^{-1} [B_k^{i'} H_k^i (B_k^1 + \sum_{j \in 2 \dots n, j \neq i} B_k^j \Psi_k^j)] ; i \in \{2 \dots n\}$$

Eventually the correctness of the rewritten equilibrium strategies $\gamma_k^{i*}(x_{k-1})$ given by (3.88) - (3.92) has to be shown.

First substitute P_k^{i*} and α_k^{i*} in (3.55) with the help of (3.106)

$$(3.88) \quad \gamma_k^{i*}(x_{k-1}) = G_k^{i*} x_{k-1} + g_k^{i*}$$

Next we deduce the feedback matrices for the leader. To do so we start by rewriting (3.56) taking consideration of (3.106)

$$G_k^{1*} = -[(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j)' H_k^1 (B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j) + R_k^{11} + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} \Psi_k^j]^{-1} \\ [(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j)' H_k^1 (A_k + \sum_{j \in 2 \dots n} B_k^j W_k^j) + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} W_k^j]$$

Using (3.93) and (3.94) yields

$$G_k^{1*} = -[\Lambda]^{-1} [\bar{B}_k' H_k^1 A_k + \bar{B}_k' H_k^1 \sum_{j \in 2 \dots n} B_k^j W_k^j + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} W_k^j]$$

Making use of (3.101) gives

$$(3.89) \quad G_k^{1*} = -[\Lambda]^{-1} [\bar{B}_k' H_k^1 A_k + \sum_{j \in 2 \dots n} \bar{D}_k^j W_k^j]$$

The constant part of the equilibrium strategy of the leader can be rewritten (considering (3.106)) in the following way

$$g_k^{1*} = -[(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j)' H_k^1 (B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j) + R_k^{11} + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} \Psi_k^j]^{-1} \\ [(B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j)' H_k^1 (s_k + \sum_{j \in 2 \dots n} B_k^j w_k^j) + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} w_k^j \\ - R_k^{11} \tilde{u}_k^{11} - \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} \tilde{u}_k^{1j} - (B_k^1 + \sum_{j \in 2 \dots n} B_k^j \Psi_k^j)' h_k^1]$$

Using (3.93) and (3.94) yields

$$g_k^{1*} = -[\Lambda]^{-1}[(B_k^{1'} + \sum_{j \in 2 \dots n} \Psi_k^{j'} B_k^{j'}) H_k^1 s_k + \bar{B}_k' H_k^1 \sum_{j \in 2 \dots n} B_k^j w_k^j + \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} w_k^j - R_k^{11} \tilde{u}_k^{11} - \sum_{j \in 2 \dots n} \Psi_k^{j'} R_k^{1j} \tilde{u}_k^{1j} - B_k^{1'} h_k^1 - \sum_{j \in 2 \dots n} \Psi_k^{j'} B_k^{j'} h_k^1]$$

Making use of (3.101), (3.104)_{i=1} and (3.105) gives

$$(3.90) \quad g_k^{1*} = -[\Lambda]^{-1}[\sum_{j \in 2 \dots n} \bar{D}_k^j w_k^j + \bar{v}_k + v_k^1]$$

Finally considering (3.106), the feedback matrices of the followers given by (3.58) and (3.59) can be rewritten as

$$(3.91) \quad G_k^{i*} = W_k^i + \Psi_k^i G_k^{1*}; i \in \{2 \dots n\}$$

$$(3.92) \quad g_k^{i*} = w_k^i + \Psi_k^i g_k^{1*}; i \in \{2 \dots n\} \quad \square$$

3.2.3 Special case: Affine-quadratic games with one leader and one follower

In this subsection, the results of Theorem (5) are specialized by reducing the number of followers from n to one.

Corollary 3 *A 2-person affine-quadratic dynamic game (cf. Def. (4)) admits a unique feedback Stackelberg equilibrium solution if*

1. $Q_k^i \geq 0$, $R_k^{ii} > 0$ and $R_k^{ij} \geq 0$ (defined for $k \in K$, $i, j \in \{1, 2\}$, $j \neq i$)

If these conditions are satisfied, the unique equilibrium strategies $\gamma_k^{i}(x_{k-1})$ ($i \in \{1, 2\}$) are given by (3.110) and the corresponding feedback Stackelberg equilibrium costs for the two players are stated in (3.120).¹⁴*

$$(3.107) \quad f_{k-1}(x_{k-1}, u_k^1, u_k^2) = A_k x_{k-1} + \sum_{j \in \{1, 2\}} B_k^j u_k^j + s_k$$

$$(3.108) \quad L^i(u^1, u^2) = \sum_{k=1}^T g_k^i(x_k, u_k^1, u_k^2, x_{k-1})$$

$$(3.109) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2} (x_k' Q_k^i x_k + \sum_{j \in \{1, 2\}} u_k^{j'} R_k^{ij} u_k^j) + \\ \frac{1}{2} (\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in \{1, 2\}} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in \{1, 2\}} \tilde{u}_k^{ij'} R_k^{ij} u_k^j$$

$$(3.110) \quad \gamma_k^{i*}(x_{k-1}) = -P_k^{i*} x_{k-1} - \alpha_k^{i*}$$

$$(3.111) \quad P_k^{1*} = [(B_k^1 + B_k^2 \bar{r}_k^2)' Z_k^1 (B_k^1 + B_k^2 \bar{r}_k^2) + R_k^{11} + \bar{r}_k^{2'} R_k^{12} \bar{r}_k^2]^{-1} \\ [(B_k^1 + B_k^2 \bar{r}_k^2)' Z_k^1 (A_k + B_k^2 W_k^2) + \bar{r}_k^{2'} R_k^{1j} W_k^2]$$

¹⁴ For all equations belonging to this theorem and its proof, $k \in K$ if nothing different is stated.

$$(3.112) \quad \alpha_k^{1*} = [(B_k^1 + B_k^2 \bar{r}_k^2)' Z_k^1 (B_k^1 + B_k^2 \bar{r}_k^2) + R_k^{11} + \bar{r}_k^{2'} R_k^{12} \bar{r}_k^2]^{-1} [(B_k^1 + B_k^2 \bar{r}_k^2)' Z_k^1 (s_k - B_k^2 w_k^2) + \bar{r}_k^{2'} R_k^{12} w_k^2 - R_k^{11} \bar{u}_k^{11} - \bar{r}_k^{2'} R_k^{12} \bar{u}_k^{12} + (B_k^1 + B_k^2 \bar{r}_k^2)' (\zeta_k^1) - Q_k^1 \bar{x}_k^1]$$

$$(3.113) \quad P_k^{2*} = (R_k^{22} + B_k^{2'} Z_k^2 B_k^2)^{-1} [B_k^{2'} Z_k^2 (A_k - B_k^1 P_k^{1*})]$$

$$(3.114) \quad \alpha_k^{2*} = (R_k^{22} + B_k^{2'} Z_k^2 B_k^2)^{-1} [B_k^{2'} (Z_k^2 (s_k - B_k^1 \alpha_k^{1*}) + \zeta_k^2 - Q_k^2 \bar{x}_k^2) - R_k^{22} \bar{u}_k^{22}]$$

$$(3.115) \quad Z_{k-1}^i = (A_k - \sum_{j \in \{1,2\}} B_k^j P_k^{j*})' Z_k^i (A_k - \sum_{j \in \{1,2\}} B_k^j P_k^{j*}) + \sum_{j \in \{1,2\}} P_k^{j*'} R_k^{ij} P_k^{j*} + Q_{k-1}^i; Z_T^i = Q_T^i$$

$$(3.116) \quad \bar{r}_k^2 = -(R_k^{22} + B_k^{2'} Z_k^2 B_k^2)^{-1} B_k^{2'} Z_k^2 B_k^1$$

$$(3.117) \quad \zeta_{k-1}^i = (A_k - \sum_{j \in \{1,2\}} B_k^j P_k^{j*})' [\zeta_k^i + Z_k^i (s_k - \sum_{j \in \{1,2\}} B_k^j \alpha_k^{j*}) - Q_k^i \bar{x}_k^i] + \sum_{j \in \{1,2\}} P_k^{j*'} R_k^{ij} (\alpha_k^{j*} + \bar{u}_k^{ij}); \zeta_T^i = 0$$

$$(3.118) \quad W_k^2 = -(R_k^{22} + B_k^{2'} Z_k^2 B_k^2)^{-1} B_k^{2'} Z_k^2 A_k$$

$$(3.119) \quad w_k^2 = -(R_k^{22} + B_k^{2'} Z_k^2 B_k^2)^{-1} [B_k^{2'} (Z_k^2 s_k + \zeta_k^2 - Q_k^2 \bar{x}_k^2) - R_k^{22} \bar{u}_k^{22}]$$

$$(3.120) \quad V^i(1, x_0) = \frac{1}{2} x_0' Z_0^i x_0 + \zeta_0^{i'} x_0 + n_0^i$$

$$\begin{aligned}
 (3.121) \quad n_{k-1}^i &= n_k^i + \frac{1}{2} \left(s_k - \sum_{j \in \{1,2\}} B_k^j \alpha_k^{j*} \right)' Z_k^i \left(s_k - \sum_{j \in \{1,2\}} B_k^j \alpha_k^{j*} \right) + \zeta_k^{i'} \\
 &\quad \left(s_k - \sum_{j \in \{1,2\}} B_k^j \alpha_k^{j*} \right) + \frac{1}{2} \sum_{j \in \{1,2\}} \alpha_k^{j*'} R_k^{ij} \alpha_k^{j*} - \tilde{x}_k^{i'} Q_k^i \left(s_k - \sum_{j \in \{1,2\}} B_k^j \alpha_k^{j*} \right) \\
 &\quad + \sum_{j \in \{1,2\}} \tilde{u}_k^{ij'} R_k^{ij} \alpha_k^{j*} + \frac{1}{2} \left(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in \{1,2\}} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij} \right); \quad n_T^i = 0
 \end{aligned}$$

PROOF:

Corollary (3) is proven in the same way as Theorem (5) taking into consideration simplifications resulting from the different number of followers. \square

Remark 10 *Special attention is drawn to the fact that the assumption about the existence of unique optimal solution sets of the systems of equations (3.58), (3.59), (3.61), (3.63) and (3.64) in Theorem (5) is not needed in Corollary (3) (and Corollary (4)), because these systems of equations degenerate to easily solvable equations in the case of only one follower.*

3.2.4 Special case: Linear-quadratic games with one leader and one follower

In this subsection, first the results of the previous subsection (3.2.3) are further specialized to a linear-quadratic 2-person game in Corollary (4) and then in Proposition (4) the specialized results are transformed into the terminology used in Corollary 7.2 in Başar and Olsder (1999, pp. 374-375)[2] to point out some serious mistakes stated there.

Corollary 4 *A 2-person linear-quadratic dynamic game (cf. Def. (4)) admits a unique feedback Stackelberg equilibrium solution if*

1. $Q_k^i \geq 0$, $R_k^{ii} > 0$ and $R_k^{ij} \geq 0$ (defined for $k \in K$, $i, j \in \{1, 2\}$, $j \neq i$)

If these conditions are satisfied, the unique equilibrium strategies $\gamma_k^{i}(x_k)$ ($i \in \{1, 2\}$) are given by (3.125) and the corresponding feedback Stackelberg equilibrium costs for the two players are stated in (3.132).¹⁵*

$$(3.122) \quad f_{k-1}(x_k, u_k^1, u_k^2) = A_k x_k + \sum_{j \in \{1, 2\}} B_k^j u_k^j$$

$$(3.123) \quad L^i(u^1, u^2) = \sum_{k=1}^T g_k^i(x_{k+1}, u_k^1(x_k), u_k^2(x_k), x_k)$$

$$(3.124) \quad g_k^i(x_{k+1}, u_k^1, u_k^2, x_k) = \frac{1}{2}(x_{k+1}' Q_{k+1}^i x_{k+1} + u_k^{i'} u_k^i + u_k^{j'} R_k^{ij} u_k^j), \quad j \neq i$$

$$(3.125) \quad \gamma_k^{j*}(x_k) = -P_k^{j*} x_k$$

$$(3.126) \quad P_k^{1*} = [(B_k^1 + B_k^2 \bar{r}_k^2)' Z_{k+1}^1 (B_k^1 + B_k^2 \bar{r}_k^2) + I + \bar{r}_k^{2'} R_k^{12} \bar{r}_k^2]^{-1} [(B_k^1 + B_k^2 \bar{r}_k^2)' Z_{k+1}^1 (A_k + B_k^2 W_k^2) + \bar{r}_k^{2'} R_k^{12} W_k^2]$$

$$(3.127) \quad P_k^{2*} = (I + B_k^{2'} Z_{k+1}^2 B_k^2)^{-1} [B_k^{2'} Z_{k+1}^2 (A_k - B_k^1 P_k^{1*})]$$

¹⁵ For all equations belonging to this theorem and its proof, $k \in K$ if nothing different is stated.

$$(3.128) \quad Z_k^i = (A_k - B_k^i P_k^{i*} - B_k^j P_k^{j*})' Z_{k+1}^i (A_k - B_k^i P_k^{i*} - B_k^j P_k^{j*}) \\ + P_k^{i*'} P_k^{i*} + P_k^{j*'} R_k^{ij} P_k^{j*} + Q_k^i; Z_{T+1}^i = Q_{T+1}^i, i, j \in \{1, 2\}, j \neq i$$

$$(3.129) \quad \bar{r}_k^2 = -(I + B_k^{2'} Z_{k+1}^2 B_k^2)^{-1} B_k^{2'} Z_{k+1}^2 B_k^1$$

$$(3.130) \quad \zeta_k^i = (A_k - \sum_{j \in \{1, 2\}} B_k^j P_k^{j*})' (\zeta_{k+1}^i) + P_k^{i*} + R_k^{ij} P_k^{j*}; \zeta_{T+1}^i = 0, j \neq i$$

$$(3.131) \quad W_k^2 = -(I + B_k^{2'} Z_{k+1}^2 B_k^2)^{-1} B_k^{2'} Z_{k+1}^2 A_k$$

$$(3.132) \quad V^i(1, x_1) = \frac{1}{2} x_1' Z_1^i x_1 + \zeta_1^{i'} x_1$$

PROOF:

For the proof of Corollary (4), the same arguments are valid as for the proof of Corollary (3). Additionally simplifications result from the modified state equation and cost functionals. \square

Remark 11 *Special attention should be paid to the observation that using a linear state equation together with a quadratic cost function without a term linearly dependent on x_{k+1} yields linear equilibrium strategies. This also holds for feedback Nash, open-loop Nash and open-loop Stackelberg games.*

Proposition 4 *The systems of equations defining the unique equilibrium strategies $\gamma_k^{i*}(x_k)$ ($i \in \{1, 2\}$) in Corollary (4) can also be written in the following way:¹⁶*

$$(3.133) \quad \gamma_k^{i*}(x_k) = -S_k^i x_k$$

¹⁶ For all equations belonging to this proposition and its proof, $k \in K$ if nothing different is stated. (3.134) is wrong in Başar and Olsder.

$$(3.134) \quad S_k^1 = [B_k^{1'}(I + Z_{k+1}^2 B_k^2 B_k^{2'})^{-1} Z_{k+1}^1 (I + B_k^2 B_k^{2'} Z_{k+1}^2)^{-1} B_k^1 + B_k^{1'} Z_{k+1}^2 B_k^2 \\ (I + B_k^{2'} Z_{k+1}^1 B_k^2)^{-1} R_k^{12} (I + B_k^{2'} Z_{k+1}^2 B_k^2)^{-1} B_k^{2'} Z_{k+1}^1 B_k^1 + I]^{-1} B_k^{1'} [(I + Z_{k+1}^2 B_k^2 B_k^{2'})^{-1} Z_{k+1}^1 \\ (I + B_k^2 B_k^{2'} Z_{k+1}^2)^{-1} + Z_{k+1}^2 B_k^{2'} (I + B_k^{2'} Z_{k+1}^1 B_k^2)^{-1} R_k^{12} (I + B_k^{2'} Z_{k+1}^2 B_k^2)^{-1} B_k^{2'} Z_{k+1}^1] A_k$$

$$(3.135) \quad S_k^2 = (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} [B_k^{2'} L_{k+1}^2 (A_k - B_k^1 S_k^1)]$$

$$(3.136) \quad L_k^i = (A_k - B_k^i S_k^i - B_k^j S_k^j)' L_{k+1}^i (A_k - B_k^i S_k^i - B_k^j S_k^j) \\ + S_k^{i'} S_k^i + S_k^{j'} R_k^{ij} S_k^j + Q_k^i; Z_{T+1}^i = Q_{T+1}^i, i, j \in \{1, 2\}, j \neq i$$

PROOF:

The proof is carried out by renaming some matrices and then showing that the relations for the equilibrium strategies γ_k^{i*} of Corollary (4) can be rewritten in the way stated above.

Let us start by renaming the feedback matrices P_k^{i*} and the matrices Z_k^i ($i \in \{1, 2\}$).

$$(3.137) \quad P_k^{i*} \triangleq S_k^i; Z_k^i \triangleq L_k^i; k \in K; i \in \{1, 2\}$$

Next we prove that the L_k^i fulfill (3.136). Taking consideration of the renaming (3.128) gives

$$(3.136) \quad L_k^i = (A_k - B_k^i S_k^i - B_k^j S_k^j)' L_{k+1}^i (A_k - B_k^i S_k^i - B_k^j S_k^j) \\ + S_k^{i'} S_k^i + S_k^{j'} R_k^{ij} S_k^j + Q_k^i; L_{T+1}^i = Q_{T+1}^i, i, j \in \{1, 2\}, j \neq i$$

Applying (3.137) to (3.127) yields

$$(3.135) \quad S_k^2 = (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} [B_k^{2'} L_{k+1}^2 (A_k - B_k^1 S_k^1)]$$

Eventually the correctness of the rewritten feedback matrix S_k^1 has to be shown. For this purpose rewrite (3.126) considering (3.137).

$$(3.138) \quad S_k^1 = [(B_k^1 + B_k^2 \bar{r}_k^2)' L_{k+1}^1 (B_k^1 + B_k^2 \bar{r}_k^2) + I \\ + \bar{r}_k^{2'} R_k^{12} \bar{r}_k^2]^{-1} [(B_k^1 + B_k^2 \bar{r}_k^2)' L_{k+1}^1 (A_k + B_k^2 W_k^2) + \bar{r}_k^{2'} R_k^{12} W_k^2]$$

Now using (3.129) and (3.131) in (3.126) (also considering (3.137)) gives

$$S_k^1 = [(B_k^1 - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1)' L_{k+1}^1 (B_k^1 - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1) + I + B_k^{1'} L_{k+1}^{2'} B_k^{2'} (I + B_k^2 L_{k+1}^i B_k^2)^{-1} R_k^{12} (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1]^{-1} \\ [(B_k^1 - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1)' L_{k+1}^1 (A_k - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 A_k) \\ + B_k^{1'} L_{k+1}^{2'} B_k^{2'} (I + B_k^2 L_{k+1}^i B_k^2)^{-1} R_k^{12} (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 A_k]$$

Finally making some algebraic manipulations and applying Lemma (4) to the four particular terms below gives

$$S_k^1 = [B_k^{1'} (I - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1)' L_{k+1}^1 (I - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1) + I + B_k^{1'} L_{k+1}^{2'} B_k^{2'} (I + B_k^2 L_{k+1}^i B_k^2)^{-1} R_k^{12} (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1]^{-1} \\ B_k^{1'} [(I - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1)' L_{k+1}^1 (I - B_k^2 (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1) \\ + L_{k+1}^{2'} B_k^{2'} (I + B_k^2 L_{k+1}^i B_k^2)^{-1} R_k^{12} (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 A_k]$$

$$(3.134) \quad S_k^1 = [B_k^{1'} (I + L_{k+1}^2 B_k^2 B_k^{2'})^{-1} L_{k+1}^1 (I + B_k^2 B_k^{2'} L_{k+1}^2)^{-1} B_k^1 + B_k^{1'} L_{k+1}^{2'} B_k^2 \\ (I + B_k^2 L_{k+1}^i B_k^2)^{-1} R_k^{12} (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2 B_k^1 + I]^{-1} B_k^{1'} [(I + L_{k+1}^2 B_k^2 B_k^{2'})^{-1} L_{k+1}^1 \\ (I + B_k^2 B_k^{2'} L_{k+1}^2)^{-1} + L_{k+1}^{2'} B_k^{2'} (I + B_k^2 L_{k+1}^i B_k^2)^{-1} R_k^{12} (I + B_k^{2'} L_{k+1}^2 B_k^2)^{-1} B_k^{2'} L_{k+1}^2] A_k \quad \square$$

4 Discrete-Time Infinite Dynamic Games with Open-Loop Information Pattern

4.1 Open-loop Nash Equilibrium Solutions

This section is devoted to the derivation of the so-called open-loop Nash equilibrium solution for affine-quadratic games. First a general result is stated about the existence of a Nash equilibrium solution in n -person discrete-time deterministic infinite dynamic games of prespecified fixed duration (cf. Def. (1)) with open-loop information pattern. Then this result is applied to affine-quadratic games.

4.1.1 Optimality conditions

This subsection contains a theorem that gives sufficient conditions for the existence of an open-loop Nash equilibrium solution and provides equations for state, control and costate vectors which have to be satisfied on the equilibrium path. Results about open-loop Nash equilibria in infinite dynamic games first appeared in continuous time in the works of Starr and Ho (1969) [15], [16] and Case (1969) [6].

Theorem 6 *For an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with open-loop information pattern let*

- $f_k(\cdot, \cdot, \dots, \cdot)$ be continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n}$ (defined for $k \in K$)
- $g_k^i(\cdot, \cdot, \dots, \cdot, \cdot)$ be continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n} \times \mathbf{R}^p$ (defined for $k \in K, i \in N$)
- $f_k(\cdot, \cdot, \dots, \cdot)$ be convex on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n}$ (defined for $k \in K$)

- $g_k^i(\cdot, \cdot, \dots, \cdot, \cdot)$ be convex on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n} \times \mathbf{R}^p$ (defined for $k \in K$, $i \in N$)
- the cost functionals be stage-additive (cf. Def. (3)).

Then the set of strategies $\{\gamma_k^{i*}(x_0) = u_k^{i*}; i \in N\}$ provides an open-loop Nash equilibrium solution. $\{x_k^*; k \in K\}$ is the corresponding state trajectory and a finite sequence of p -dimensional costate vectors $\{p_1^i, \dots, p_K^i\}$ (defined for $i \in N$) exists so that the following relations are satisfied:

$$(4.1) \quad x_k^* = f_{k-1}(x_{k-1}^*, u_k^{1*}, \dots, u_k^{n*}), \quad x_0^* = x_0$$

$$(4.2) \quad \nabla_{u_k^i} H_k^i(p_k^i, u_k^{1*}, \dots, u_k^{i-1*}, u_k^i, u_k^{i+1*}, \dots, u_k^{n*}, x_{k-1}^*) = 0$$

$$(4.3) \quad p_k^i = \frac{\partial}{\partial x_k} f_k(x_k^*, u_{k+1}^{1*}, \dots, u_{k+1}^{n*})' [p_{k+1}^i + (\frac{\partial}{\partial x_{k+1}} g_{k+1}^i(x_{k+1}^*, u_{k+1}^{1*}, \dots, u_{k+1}^{n*}, x_k^*))'] + [\frac{\partial}{\partial x_k} g_{k+1}^i(x_{k+1}^*, u_{k+1}^{1*}, \dots, u_{k+1}^{n*}, x_k^*)]'; \quad p_T^i = 0$$

where

$$(4.4) \quad H_k^i(p_k^i, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*) \hat{=} g_k^i(f_{k-1}(x_{k-1}^*, u_k^{1*}, \dots, u_k^{n*}), u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*) + p_k^{i'} f_{k-1}(x_{k-1}^*, u_k^{1*}, \dots, u_k^{n*})$$

Every such equilibrium solution is weakly time consistent.

PROOF:

Theorem (4) can be proven in the same way as Theorem 6.1 in Başar and Olsder (1999, pp. 267-268)[2], bearing in mind that in the above theorem we additionally assume (implicitly) the convexity of the cost functionals. That is why we apply Theorem (1) (instead of Theorem 5.5 in Başar and Olsder (1999, p. 246)[2]) to the standard optimal control problem for player i which emerges in the course of the

derivation. That is why not only necessary but necessary and sufficient conditions are stated in the above theorem. \square

4.1.2 Results for affine-quadratic games with arbitrarily many players

In the following the results of Theorem (6) are applied to an affine-quadratic dynamic game with arbitrarily many players. Theorem (7), which is an extension (concerning the cost functionals) of Theorem 6.2 in Başar and Olsder (1999, pp. 269-271)[2], presents equilibrium equations that can easily be used for an algorithmic disintegration of the given Nash game.

Furthermore in Proposition (5) the equivalence of the equations of Theorem (7) with terminologically different equations is shown.

Theorem 7 *An n -person affine-quadratic dynamic game (cf. Def. (4)) admits a unique open-loop Nash equilibrium solution if*

- $Q_k^i \geq 0, R_k^{ii} > 0$ (defined for $k \in K, i \in N$).
- $(I + \sum_{j \in N} B_k^j (R_k^{jj})^{-1} B_k^{j'} M_k^j)^{-1}$ (defined for $k \in K$) exist.

If these conditions are satisfied, the unique equilibrium strategies are given by (4.13), where the associated state trajectory x_{k+1}^ is given by (4.8).¹⁷*

$$(4.5) \quad f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k; k \in K$$

$$(4.6) \quad L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$$

$$(4.7) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2} (x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) + \frac{1}{2} (\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j; k \in K$$

$$(4.8) \quad x_{k+1}^* = \Phi_k x_k^* + \phi_k$$

¹⁷ For all equations belonging to this theorem and its proof, $i \in N$ and $k \in \{0, \dots, T-1\}$ if nothing different is stated.

$$(4.9) \quad \Phi_k = (I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} M_{k+1}^j)^{-1} A_{k+1}$$

$$(4.10) \quad \phi_k = (I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} M_{k+1}^j)^{-1} (s_{k+1} - \sum_{j \in N} B_{k+1}^j ((R_{k+1}^{jj})^{-1} B_{k+1}^{j'} (m_{k+1}^j - Q_{k+1}^j \tilde{x}_{k+1}^{j'}) - \tilde{u}_{k+1}^{jj}))$$

$$(4.11) \quad M_k^i = Q_k^i + A_{k+1}' M_{k+1}^i \Phi_k ; M_T^i = Q_T^i$$

$$(4.12) \quad m_k^i = A_{k+1}' [M_{k+1}^i \phi_k + m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^i] ; m_T^i = 0$$

$$(4.13) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = -P_{k+1}^{i*} x_k^* - \alpha_{k+1}^{i*}$$

$$(4.14) \quad P_{k+1}^{i*} = (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^i \Phi_k$$

$$(4.15) \quad \alpha_{k+1}^i = (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (M_{k+1}^i \phi_k + m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^{i'}) + \tilde{u}_{k+1}^{ii}$$

PROOF:

Theorem (6) can be applied to the given affine-quadratic game, since all conditions are satisfied for the given state equation (4.5) and cost functionals (4.6). Furthermore g_k^i is strictly convex in u_k^i . This can be seen by applying Corollary (1) to (4.16). Therefore there has to be a unique optimal equilibrium solution.

$$(4.16) \quad \frac{\partial^2}{\partial u_i^2} g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = B_k^{i'} Q_k^i B_k^i + R_k^{ii}$$

To obtain relations which satisfy this unique solution, we have to adapt (4.1) - (4.4) to the given state equation and cost functionals. This yields

$$(4.17) \quad H_k^i = \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) \\ - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j + p_k^{i'} (A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k)$$

$$(4.18) \quad p_k^i = A_{k+1}' [p_{k+1} + Q_{k+1}^i (x_{k+1}^* - \tilde{x}_{k+1}^i)] ; p_T^i = 0$$

$$\frac{\partial}{\partial u_i}(4.17) = 0 \Rightarrow B_k^{i'} Q_k^i x_k^* + R_k^{ii} u_k^{i*} - B_k^{i'} Q_k^i \tilde{x}_k^{i'} - R_k^{ii} \tilde{u}_k^{ii} + B_k^{i'} p_k^i = 0$$

$$(4.19) \quad u_k^{i*} = -(R_k^{ii})^{-1} B_k^{i'} (Q_k^i (x_k^* - \tilde{x}_k^{i'}) + p_k^i) + \tilde{u}_k^{ii}$$

$$(4.20) \quad x_k^* = A_k x_{k-1}^* + \sum_{j \in N} B_k^j u_k^{j*} + s_k ; x_0^* = x_0$$

In the following induction argument we will give proof that (4.21) is valid and the recursive relations for M_k^i and m_k^i (stated in the above theorem) are correct.

$$(4.21) \quad p_k^i = (M_k^i - Q_k^i) x_k^* + m_k^i$$

Basis:

The induction starts at $k = T$. First we make use of the general optimality conditions for p_k^i at stage T .

$$(4.18)_{k=T} \quad p_T^i = 0$$

Now we can substitute the p_T^i with functions affinely dependent on x_T .

$$(4.21)_{k=T} \quad p_T^i = (M_T^i - Q_T^i)x_T^* + m_T^i$$

$$(M_T^i - Q_T^i)x_T^* + m_T^i = 0$$

Comparing coefficients gives

$$(4.11)_{k=T} \quad M_T^i = Q_T^i$$

$$(4.12)_{k=T} \quad m_T^i = 0$$

Inductive step:

As an induction hypothesis, the system of equations (4.21) is assumed to be true at stage $l+1$. Now we have to prove that this system of equations is fulfilled at stage l and determines the corresponding recursive relations for M_l^i and m_l^i .

$$(4.21)_{k=l+1} \quad p_{l+1}^i = (M_{l+1}^i - Q_{(l+1)}^i)x_{l+1}^* + m_{l+1}^i$$

First the induction hypothesis is used in the general optimality conditions for p_k^i at stage l .

$$(4.18)_{k=l} \quad p_l^i = A'_{l+1}[p_{l+1} + Q_{l+1}^i(x_{l+1}^* - \tilde{x}_{l+1}^i)]$$

$$p_l^i = A'_{l+1}[(M_{l+1}^i - Q_{l+1}^i)x_{l+1}^* + m_{l+1}^i + Q_{l+1}^i(x_{l+1}^* - \tilde{x}_{l+1}^i)]$$

$$p_l^i = A'_{l+1}[M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]$$

To complete the inductive step we have to prove that the p_l^i can be written as affine functions of x_l . Therefore an interrelation between x_l and x_{l+1} , which does not depend on the controls of the players nor on the costate variables p_{l+1}^i , has to be deduced. To do so, first we have to substitute u_{l+1}^{i*} in the equation stated below for the evolution of the optimal state vector x_{l+1}^* by terms which are affine in x_l and x_{l+1} and furthermore only contain M_{l+1}^i , m_{l+1}^i and matrices and vectors given by the game definition.

$$(4.20)_{k=l+1} \quad x_{l+1}^* = A_{l+1}x_l^* + \sum_{j \in N} B_{l+1}^j u_{l+1}^{j*} + s_{l+1}$$

In the first instance the optimality condition for u_{l+1}^{i*} can be rewritten with the help of (4.21) $_{k=l+1}$, which are the induction hypotheses for p_{l+1}^i .

$$(4.19)_{k=l+1} \quad u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (Q_{l+1}^i (x_{l+1}^* - \tilde{x}_{l+1}^{i'}) + p_{l+1}^i) + \tilde{u}_{l+1}^{ii}$$

$$\begin{aligned} u_{l+1}^{i*} = & -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (Q_{l+1}^i (x_{l+1}^* - \tilde{x}_{l+1}^{i'}) \\ & + (M_{l+1}^i - Q_{l+1}^i)x_{l+1}^* + m_{l+1}^i) + \tilde{u}_{l+1}^{ii} \end{aligned}$$

$$u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}$$

Now the control variables can be replaced in the optimal state equation by terms affine in x_{l+1} .

$$\begin{aligned} x_{l+1}^* = A_{l+1} x_l^* - \sum_{j \in N} B_{l+1}^j ((R_{l+1}^{jj})^{-1} [B_{l+1}^{j'} (M_{l+1}^j x_{l+1}^* \\ + m_{l+1}^j - Q_{l+1}^j \tilde{x}_{l+1}^{j'})] + \tilde{u}_{l+1}^{jj}) + s_{l+1} \end{aligned}$$

Making x_{l+1} explicit gives

$$\begin{aligned} x_{l+1}^* = (I + \sum_{j \in N} B_{l+1}^j (R_{l+1}^{jj})^{-1} B_{l+1}^{j'} M_{l+1}^j)^{-1} [A_{l+1} x_l^* \\ - \sum_{j \in N} B_{l+1}^j ((R_{l+1}^{jj})^{-1} B_{l+1}^{j'} (m_{l+1}^j - Q_{l+1}^j \tilde{x}_{l+1}^{j'}) - \tilde{u}_{l+1}^{jj}) + s_{l+1}] \end{aligned}$$

The structure of the above equation justifies the following substitution

$$(4.8)_{k=l} \quad x_{l+1}^* = \Phi_l x_l^* + \phi_l$$

$$\begin{aligned} (4.22) \quad \Phi_l x_l^* + \phi_l = (I + \sum_{j \in N} B_{l+1}^j (R_{l+1}^{jj})^{-1} B_{l+1}^{j'} M_{l+1}^j)^{-1} [A_{l+1} x_l^* \\ - \sum_{j \in N} B_{l+1}^j ((R_{l+1}^{jj})^{-1} B_{l+1}^{j'} (m_{l+1}^j - Q_{l+1}^j \tilde{x}_{l+1}^{j'}) - \tilde{u}_{l+1}^{jj}) + s_{l+1}] \end{aligned}$$

By comparing coefficients it follows that

$$(4.22)_{x_l^*} = (4.9)_{k=l} \quad \Phi_l = (I + \sum_{j \in N} B_{l+1}^j (R_{l+1}^{jj})^{-1} B_{l+1}^{j'} M_{l+1}^j)^{-1} A_{l+1}$$

$$(4.22)_{const.} = (4.10)_{k=l} \quad \phi_l = (I + \sum_{j \in N} B_{l+1}^j (R_{l+1}^{jj})^{-1} B_{l+1}^{j'} M_{l+1}^j)^{-1} (s_{l+1} - \sum_{j \in N} B_{l+1}^j ((R_{l+1}^{jj})^{-1} B_{l+1}^{j'} (m_{l+1}^j - Q_{l+1}^j \tilde{x}_{l+1}^{j'}) - \tilde{u}_{l+1}^{jj}))$$

The above relation between x_{l+1} and x_l , given by equation $(4.8)_{k=l}$, can be used to finish the inductive step for p_l^i .

$$p_l^i = A_{l+1}' [M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]$$

$$p_l^i = A_{l+1}' [M_{l+1}^i (\Phi_l x_l^* + \phi_l) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]$$

The structure of the above equations justifies the following substitutions

$$(4.21)_{k=l} \quad p_l^i = (M_l^i - Q_l^i) x_l^* + m_l^i$$

$$(4.23) \quad (M_l^i - Q_l^i) x_l^* + m_l^i = A_{l+1}' [M_{l+1}^i (\Phi_l x_l^* + \phi_l) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]$$

Comparing coefficients gives

$$(4.23)_{x_l^*} = (4.11)_{k=l} \quad M_l^i = Q_l^i + A_{l+1}' M_{l+1}^i \Phi_l$$

$$(4.23)_{const.} = (4.12)_{k=l} \quad m_l^i = A_{l+1}' [M_{l+1}^i \phi_l + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]$$

At this point the inductive step and hence the induction argument is completed, but we try to transform u_{l+1}^{i*} so that their evolution depends affinely on x_l^* and therefore,

their algorithmic computation is straightforward. This is done by using (4.8)_{k=l} in the equation deduced above for u_{l+1}^{i*} .

$$u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}$$

$$u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i (\Phi_l x_l^* + \phi_l) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}$$

The structure of the above equations justifies the following substitutions

$$(4.13)_{k=l} \quad \gamma_{l+1}^{i*}(x_0) = u_{l+1}^{i*} = -P_{l+1}^{i*} x_l^* - \alpha_{l+1}^{i*}$$

$$(4.24) \quad -P_{l+1}^{i*} x_l^* - \alpha_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i (\Phi_l x_l^* + \phi_l) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}$$

Comparing coefficients it follows that

$$(4.24)_{x_l^*} = (4.14)_{k=l} \quad P_{l+1}^{i*} = (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^i \Phi_l$$

$$(4.24)_{const.} = (4.15)_{k=l} \quad \alpha_{l+1}^i = (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i \phi_l + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} \quad \square$$

Remark 12 By making the induction hypothesis that p_k^i is dependent on x_k (instead of x_{k+1} in Başar and Olsder (1999, pp.269 - 271)[2]) the number of algebraic manipulations and substitutions is decreased to less than half and also the complexity and length of the equations is considerably reduced.

Remark 13 *Special attention should be paid to the fact that the induction argument only runs over the costate vectors p_k^i , and the general relations for x_k and u_k^i given in (4.8) and (4.13) arise from the combination of the general optimality conditions for x_k and u_k^i given in (4.20) and (4.19) (which have to hold true at each stage of the game) and the induction hypothesis for p_k^i . In contrast the proof in Başar and Olsder (1999, pp.269 - 271)[2]) looks as though an induction for p_k^i , x_k and u_k^i is being made with the basis at the last stage (which is not allowed because the basis for x_k and u_k^i cannot be verified (in a correct way) at the last stage), but not using the induction hypothesis for x_k and u_k^i in the inductive step.*

Remark 14 *To solve the Nash game algorithmically, the following order of application of the equations of Theorem (7) is advisable ($i \in N$):*

1. M_T^i, m_T^i
2. For k running backward from $T-1$ to 0
 - a) Φ_k, ϕ_k
 - b) M_k^i, m_k^i
3. x_0^*
4. For k running forward from 0 to $T-1$
 - a) $P_{k+1}^i, \alpha_{k+1}^i$
 - b) u_{k+1}^{i*}
 - c) x_{k+1}^*
 - d) $g_{k+1}^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$
5. $L^i(x_0, u^1, \dots, u^n)$

Proposition 5 ¹⁸ *The systems of equations defining the unique equilibrium strategies $\gamma_{k+1}^*(x_0)$ and the associated state trajectory x_{k+1}^* in Theorem (7) can also be written in the following way:*¹⁹

$$(4.25) \quad x_{k+1}^* = (\Lambda_{k+1})^{-1} (A_{k+1} x_k^* + \eta_{k+1})$$

$$(4.26) \quad \Lambda_{k+1} = I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} H_{k+1}^j$$

$$(4.27) \quad \eta_{k+1} = s_{k+1} + \sum_{j \in N} B_{k+1}^j (\tilde{u}_{k+1}^{jj} - (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} h_{k+1}^j)$$

$$(4.28) \quad H_k^i = Q_k^i + A_{k+1}' H_{k+1}^i (\Lambda_{k+1})^{-1} A_{k+1} ; H_T^i = Q_T^i$$

$$(4.29) \quad h_k^i = -Q_k^i \tilde{x}_k^i + A_{k+1}' [H_{k+1}^i (\Lambda_{k+1})^{-1} \eta_{k+1} + h_{k+1}^i] ; h_T^i = -Q_T^i \tilde{x}_T^i$$

$$(4.30) \quad \gamma_{k+1}^*(x_0) = u_{k+1}^{i*} = \tilde{u}_{k+1}^{ii} - (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (H_{k+1}^i x_{k+1}^* + h_{k+1}^i)$$

PROOF:

The proof is done by renaming the costate matrices and then showing that the relations for the costate matrices and the optimal state and control vectors of Theorem (7) can be rewritten in the way stated above.

Let us start by renaming the costate matrices M_k^i and m_k^i .

$$(4.31) \quad M_k^i \triangleq H_k^i ; m_k^i \triangleq h_k^i + Q_k^i \tilde{x}_k^i$$

¹⁸ In this proposition we rewrite the equilibrium equations in a notation that was used at our department in the past to enable comparison.

¹⁹ For all equations belonging to this proposition and its proof, $k \in K$ and $i \in N$ if nothing different is stated.

Next we prove that the costate matrices fulfill (4.28) and (4.29) respectively.

Taking the renaming into consideration (4.11) gives

$$H_k^i = Q_k^i + A'_{k+1} H_{k+1}^i \Phi_k$$

Substituting Φ_k with the help of (4.9) (and considering (4.31)) leads to

$$H_k^i = Q_k^i + A'_{k+1} H_{k+1}^i \left(I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} H_{k+1}^j \right)^{-1} A_{k+1}$$

Making use of (4.26) yields

$$(4.28) \quad H_k^i = Q_k^i + A'_{k+1} H_{k+1}^i (\Lambda_{k+1})^{-1} A_{k+1}$$

Now we show the correctness of (4.29). To do this we start at stage T and use (4.12) and (4.31) to get

$$0 = m_T^i = h_T^i + Q_T^i \tilde{x}_T^i$$

$$(4.29)_{k=T} \quad h_T^i = -Q_T^i \tilde{x}_T^i$$

For the general stage k rewrite (4.12) taking consideration of (4.31)

$$h_k^i + Q_k^i \tilde{x}_k^i = A'_{k+1} [H_{k+1}^i \phi_k + h_{k+1}^i]$$

Substituting ϕ_k with the help of (4.10) (and considering (4.31)) yields

$$h_k^i = -Q_k^i \tilde{x}_k^i + A_{k+1}' [H_{k+1}^i (I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} H_{k+1}^j)^{-1} (s_{k+1} - \sum_{j \in N} B_{k+1}^j ((R_{k+1}^{jj})^{-1} B_{k+1}^{j'} h_{k+1}^j - \tilde{u}_{k+1}^{jj})) + h_{k+1}^i]$$

Eventually using (4.26) and (4.27) gives

$$(4.29) \quad h_k^i = -Q_k^i \tilde{x}_k^i + A_{k+1}' [H_{k+1}^i (\Lambda_{k+1})^{-1} \eta_{k+1} + h_{k+1}^i]$$

Next we show the correctness of (4.25). To do this we start with equation (4.8) and substitute Φ_k and ϕ_k (and also consider (4.31))

$$x_{k+1}^* = (I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} H_{k+1}^j)^{-1} A_{k+1} x_k^* + (I + \sum_{j \in N} B_{k+1}^j (R_{k+1}^{jj})^{-1} B_{k+1}^{j'} H_{k+1}^j)^{-1} (s_{k+1} - \sum_{j \in N} B_{k+1}^j ((R_{k+1}^{jj})^{-1} B_{k+1}^{j'} h_{k+1}^j - \tilde{u}_{k+1}^{jj}))$$

Making use of (4.26) and (4.27) we get

$$(4.25) \quad x_{k+1}^* = (\Lambda_{k+1})^{-1} (A_{k+1} x_k^* + \eta_{k+1})$$

Eventually the correctness of the rewritten equilibrium strategies u_{k+1}^{i*} , given by (4.30), has to be shown.

First substituting P_{k+1}^i and α_{k+1}^i in (4.13) with the help of (4.14) and (4.15) (and also considering (4.31)) gives

$$u_{k+1}^{i*} = -(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} H_{k+1}^i \Phi_k x_k^* - (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (H_{k+1}^i \phi_k + h_{k+1}^i) + \tilde{u}_{k+1}^{ii}$$

Simplifying the above equation leads to

$$u_{k+1}^{i*} = \tilde{u}_{k+1}^{ii} - (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} H_{k+1}^i (\Phi_k x_k^* + \phi_k + h_{k+1}^i)$$

Finally using (4.8) yields

$$(4.30) \quad u_{k+1}^{i*} = \tilde{u}_{k+1}^{ii} - (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} H_{k+1}^i (x_{k+1}^* + h_{k+1}^i) \quad \square$$

4.2 Open-loop Stackelberg Equilibrium Solutions

This section is devoted to the derivation of the so-called open-loop Stackelberg equilibrium solution with one leader and arbitrarily many followers for affine-quadratic games. First a general result is stated about the existence and uniqueness of a Stackelberg equilibrium solution with one leader and arbitrarily many followers in n -person discrete-time deterministic infinite dynamic games of prespecified fixed duration (cf. Def. (1)) with open-loop information pattern. Then this result is applied to affine-quadratic games. First a proof geared to the one indicated in Başar and Olsder (1999, p. 372)[2] is presented. But this proof "produces" a hardly algorithmically solvable system of equilibrium equations. Therefore another way of deriving the equilibrium solution for affine-quadratic games is presented giving us a system of equilibrium equations that can easily be used for an algorithmic disintegration of the given Stackelberg game.

4.2.1 Optimality conditions

The following theorem gives sufficient conditions for the existence of an open-loop Stackelberg equilibrium solution with one leader and arbitrarily many followers and provides equations for state, control, costate and cocontrol vectors, which have to be satisfied on the equilibrium path. Results about open-loop Stackelberg equilibria in infinite dynamic games first appeared in continuous time in the works of Chen and Cruz (1972) [7] and Simaan and Cruz (1973) [13], [14].

Theorem 8 *For an n -person discrete-time deterministic infinite dynamic game of prespecified fixed duration (cf. Def. (1)) with open-loop information pattern let*

- $f_k(x_{k-1}, \cdot, u_k^2, \dots, u_k^n)$ be continuously differentiable on \mathbf{R}^{m_1} (defined for $k \in K$)
- $f_k(\cdot, u_k^1, \cdot, \dots, \cdot)$ be twice continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^{m_2} \times \dots \times \mathbf{R}^{m_n}$ (defined for $k \in K$)
- $g_k^1(\cdot, \cdot, \dots, \cdot, \cdot)$ be continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n} \times \mathbf{R}^p$ (defined for $k \in K$)
- $g_k^i(x_k, \cdot, u_k^2, \dots, u_k^n, x_{k-1})$ be continuously differentiable on \mathbf{R}^{m_1} (defined for $k \in K, i \in \{2 \dots n\}$)

- $g_k^i(\cdot, u_k^1, \cdot, \dots, \cdot, \cdot)$ be twice continuously differentiable on $\mathbf{R}^p \times \mathbf{R}^{m_2} \times \dots \times \mathbf{R}^{m_n} \times \mathbf{R}^p$ (defined for $k \in K, i \in \{2 \dots n\}$)
- $f_k(\cdot, \cdot, \dots, \cdot)$ be convex on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n}$ (defined for $k \in K$)
- $g_k^i(\cdot, \cdot, \dots, \cdot, \cdot)$ be strictly convex on $\mathbf{R}^p \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n} \times \mathbf{R}^p$ (defined for $k \in K, i \in N$)
- the cost functionals be stage-additive (cf. Def. (3)).

Then the set of strategies $\{\gamma_k^{i*}(x_0) = u^{i*}; i \in N\}$ is unique and provides an open-loop Stackelberg equilibrium solution with **P1** as the leader and **P2** ... **Pn** as followers. Furthermore the corresponding state trajectory $\{x_k^*; k \in K\}$, the m -dimensional co-control vectors of the leader $\{v_1^i, \dots, v_{T-1}^i; i \in \{2 \dots n\}\}$ and the p -dimensional costate vectors $\{\lambda_1, \dots, \lambda_T, \mu_1^i, \dots, \mu_T^i, p_1^{i*}, \dots, p_T^{i*}\}$ (defined for $i \in \{2 \dots n\}$) exist such that the following relations are satisfied:

$$(4.32) \quad x_k^* = f_{k-1}(x_{k-1}^*, u_k^{1*}, \dots, u_k^{n*}), \quad x_0^* = x_0$$

$$(4.33) \quad \nabla_{u_k^1} H_k^1(\lambda_k, \mu_{k-1}^2, \dots, \mu_{k-1}^n, v_{k-1}^2, \dots, v_{k-1}^n, p_k^{2*}, \dots, p_k^{n*}, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*) = 0$$

$$(4.34) \quad \nabla_{u_k^i} H_k^i(\lambda_k, \mu_k^2, \dots, \mu_{k-1}^2, \dots, \mu_{k-1}^n, v_{k-1}^2, \dots, v_{k-1}^n, p_k^{2*}, \dots, p_k^{n*}, u_k^{1*}, u_k^{2*}, \dots, u_k^{n*}, x_{k-1}^*) = 0; i \in \{2 \dots n\}$$

$$(4.35) \quad \lambda_{k-1}^i = \frac{\partial}{\partial x_{k-1}} H_k^i(\lambda_k, \mu_{k-1}^2, \dots, \mu_{k-1}^n, v_{k-1}^2, \dots, v_{k-1}^n, p_k^{2*}, \dots, p_k^{n*}, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*); \lambda_T^i = 0$$

$$(4.36) \quad \mu_k^{i'} = \frac{\partial}{\partial p_k} H_k^1(\lambda_k, \mu_{k-1}^2, \dots, \mu_{k-1}^n, v_{k-1}^2, \dots, v_{k-1}^n, \\ p_k^{2*}, \dots, p_k^{n*}, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*) ; \mu_0^i = 0 ; i \in \{2 \dots n\}$$

$$(4.37) \quad \nabla_{u_k^i} H_k^i(p_k^{i*}, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*) = 0 ; i \in \{2 \dots n\}$$

$$(4.38) \quad p_{k-1}^{i*} = F_{k-1}^i(x_{k-1}^*, u_k^{1*}, \dots, u_k^{n*}, p_k^i) ; p_T^{i*} = 0 ; i \in \{2 \dots n\}$$

where

$$(4.39) \quad H_k^1(\lambda_k, \mu_{k-1}^2, \dots, \mu_{k-1}^n, v_{k-1}^2, \dots, v_{k-1}^n, p_k^2, \dots, p_k^n, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*) \triangleq \\ g_k^1(f_k(x_{k-1}, u_k^1, \dots, u_k^n), u_k^1, \dots, u_k^n, x_{k-1}) + \lambda_k' f_k(x_{k-1}, u_k^1, \dots, u_k^n) + \\ \sum_{j \in \{2 \dots n\}} \mu_{k-1}^{j'} F_{k-1}^j(x_{k-1}, u_k^1, \dots, u_k^n, p_k^j) + \sum_{j \in \{2 \dots n\}} v_{k-1}^{j'} (\nabla_{u_k^j} H_k^j(p_k^j, u_k^{1*}, \dots, u_k^{n*}, x_{k-1}^*))$$

$$(4.40) \quad F_k^i(x_k, u_{k+1}^1, \dots, u_{k+1}^n, p_{k+1}^i) \triangleq \frac{\partial}{\partial x_k} f_k(x_k, u_{k+1}^1, \dots, u_{k+1}^n)' [p_{k+1}^i \\ + (\frac{\partial}{\partial x_{k+1}} g_{k+1}^i(x_{k+1}, u_{k+1}^1, \dots, u_{k+1}^n, x_k))'] + [\frac{\partial}{\partial x_k} g_{k+1}^i(x_{k+1}, u_{k+1}^1, \dots, u_{k+1}^n, x_k)]' ; i \in \{2 \dots n\}$$

$$(4.41) \quad H_k^i(p_k^i, u_k^1, \dots, u_k^n, x_{k-1}) \triangleq g_k^i(f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n), \\ u_k^1, \dots, u_k^n, x_{k-1}) + p_k^{i'} f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) ; i \in \{2 \dots n\}$$

PROOF:

Theorem (8) can be proven in the same way as Theorem (6), bearing in mind that the leader additionally accounts for the influence of his strategy on the followers' strategies when minimizing his cost functional. Thus the minimization problem of the leader is equivalent to a finite dimensional nonlinear programming problem.

The solution vectors of this problem are stated in (4.33) - (4.36) and a derivation of this solution can be found e.g. in Canon et al. (1970, p. 51) [5]. \square

4.2.2 The "interwoven-inductions-results" for affine-quadratic games with one leader and arbitrarily many followers

In the following, the results of Theorem (8) are applied to an affine-quadratic dynamic game with one leader and arbitrarily many followers. Theorem (9) is a generalization of Corollary 7.1 in Başar and Olsder (1999, pp. 371-372) [2]. On the one hand a more general state equation and more general cost functionals are considered and on the other hand the number of followers is extended from one to arbitrarily many.

The structure of the proof is geared to the one indicated in Başar and Olsder (1999, p. 372) [2] where two induction arguments are interwoven. The induction for μ_k^i ($i \in \{2 \dots n\}$, being the costate vectors of the leader, which are associated with the costate vectors of the followers) runs forward in time from $k=0$ to $k=T-1$ and the induction for p_k^i and λ_k ($i \in \{2 \dots n\}$, being the costate vectors of the followers and the costate vectors of the leader) runs backward in time from $k=T$ to $k=1$. In the inductive step the induction hypotheses of the two inductions are used together.

This fact causes severe problems if we intend to use the obtained equilibrium equations for an algorithmic disintegration of the game because the evolution of C_k^i and c_k^i ($i \in \{2 \dots n\}$, $\mu_k^i = C_k^i x_k + c_k^i$) is dependent on M_{k+1}^i , m_{k+1}^i , L_{k+1} and l_{k+1} ($i \in \{2 \dots n\}$, $p_k^i = M_k^i x_k + m_k^i$, $\lambda_k^i = L_k x_k + l_k$) and the evolution of M_{k+1}^i , m_{k+1}^i , L_{k+1} and l_{k+1} is dependent on C_k^i and c_k^i . But C_k^i , c_k^i , M_{k+1}^i , m_{k+1}^i , L_{k+1} and l_{k+1} are needed to determine Φ_k and ϕ_k , which define the evolution of the state variable and the control variables. This means that to solve the game we have to determine C_k^i (determining c_k^i is even more complicated) as a function of $(L_1, \dots, L_k, M_1^i, \dots, M_k^i)$ for each $k \in \{2, \dots, T\}$ (which is all but impossible even for a very small number of stages) and then substitute C_k^i when determining L_{k+1} and M_{k+1}^i backwards in time.

To overcome these intractabilities, another proof structure has to be used to solve the dynamic, affine-quadratic Stackelberg game. In subsection (4.2.5) a theorem is stated which is proven by showing that p_k^i and λ_k ($i \in \{2 \dots n\}$) can be determined as functions linearly dependent on x_k and μ_k^i ($i \in \{2 \dots n\}$). Solving the dynamic,

affine-quadratic Stackelberg game in that way yields equilibrium equations that can easily be used for an algorithmic disintegration of that game.

Theorem 9 *An n -person affine-quadratic dynamic game (cf. Def. (4)) admits a unique open-loop Stackelberg equilibrium solution with one leader and arbitrarily many followers if*

- $Q_k^i \geq 0, R_k^{ii} > 0$ (defined for $k \in K, i \in N$).
- $(I - B_{k+1}^1 W_{k+1}^1 - \sum_{j \in \{2 \dots n\}} B_{k+1}^j T_{k+1}^j)^{-1}$ and $(A_{k+1} + B_{k+1}^1 W_{k+1}^0)^{-1}$ (defined for $k \in K$) exist.
- (4.53), (4.54) and (4.55) admit unique solutions N_k^{1i}, N_k^{0i} and n_k^i (defined for $k \in K, i \in \{2, \dots, n\}$)

If these conditions are satisfied, the unique equilibrium strategies are given by (4.62), where the associated state trajectory x_{k+1}^* is given by (4.45).²⁰

$$(4.42) \quad f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k; k \in K$$

$$(4.43) \quad L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$$

$$(4.44) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) \\ + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j; k \in K$$

$$(4.45) \quad x_{k+1}^* = \Phi_k x_k^* + \phi_k; x_0^* = x_0$$

$$(4.46) \quad \Phi_k = (I - B_{k+1}^1 W_{k+1}^1 - \sum_{j \in \{2 \dots n\}} B_{k+1}^j T_{k+1}^j)^{-1} (A_{k+1} + B_{k+1}^1 W_{k+1}^0)$$

²⁰ For all equations belonging to this theorem and its proof, $i \in N$ and $k \in \{0, \dots, T-1\}$ if nothing different is stated.

$$(4.47) \quad \phi_k = (I - B_{k+1}^1 W_{k+1}^1 - \sum_{j \in \{2 \dots n\}} B_{k+1}^j T_{k+1}^j) - 1 \\ (B_{k+1}^1 w_{k+1} + \sum_{j \in \{2 \dots n\}} B_{k+1}^j t_{k+1}^j + s_{k+1})$$

$$(4.48) \quad W_{k+1}^1 = -(R_{k+1}^{11})^{-1} (B_{k+1}^{1'} L_{k+1} + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j B_{k+1}^j N_k^{1j})$$

$$(4.49) \quad W_{k+1}^0 = -(R_{k+1}^{11})^{-1} (\sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j A_{k+1} C_k^j \\ + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j B_{k+1}^j N_k^{0j})$$

$$(4.50) \quad w_{k+1} = -(R_{k+1}^{11})^{-1} (-B_{k+1}^{1'} Q_{k+1}^1 \tilde{x}_{k+1}^1 + B_{k+1}^{1'} l_{k+1} \\ + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j A_{k+1} c_k^j + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j B_{k+1}^j n_k^j) + \tilde{u}_{k+1}^{11}$$

$$(4.51) \quad T_{k+1}^i = -(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^i ; i \in \{2 \dots n\}$$

$$(4.52) \quad t_{k+1}^i = -(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^{i'}) + \tilde{u}_{k+1}^{ii} ; i \in \{2 \dots n\}$$

$$(4.53) \quad -R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^i + B_{k+1}^{i'} L_{k+1} + (B_{k+1}^{i'} Q_{k+1}^i B_{k+1}^i \\ + R_{k+1}^{ii}) N_k^{1i} + \sum_{j \in 2 \dots n, j \neq i} B_{k+1}^{i'} Q_{k+1}^j B_{k+1}^j N_k^{1j} = 0 ; i \in \{2 \dots n\}$$

$$(4.54) \quad \sum_{j \in 2 \dots n} B_{k+1}^{i'} Q_{k+1}^j A_{k+1} C_k^j + (B_{k+1}^{i'} Q_{k+1}^i B_{k+1}^i + R_{k+1}^{ii}) N_k^{0i} \\ + \sum_{j \in 2 \dots n, j \neq i} B_{k+1}^{i'} Q_{k+1}^j B_{k+1}^j N_k^{0j} = 0; i \in \{2 \dots n\}$$

$$(4.55) \quad -B_{k+1}^{i'} Q_{k+1}^1 \tilde{x}_{k+1}^{i*} + R_{k+1}^{1i} (-(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^{i'}) + \tilde{u}_{k+1}^{ii} \\ - \tilde{u}_{k+1}^{1i}) + B_{k+1}^{i'} l_{k+1} + \sum_{j \in 2 \dots n} B_{k+1}^{i'} Q_{k+1}^j A_{k+1} c_k^j + (B_{k+1}^{i'} Q_{k+1}^i B_{k+1}^i \\ + R_{k+1}^{ii}) n_k^i + \sum_{j \in 2 \dots n, j \neq i} B_{k+1}^{i'} Q_{k+1}^j B_{k+1}^j n_k^j = 0; i \in \{2 \dots n\}$$

$$(4.56) \quad M_k^i = Q_k^i + A_{k+1}' M_{k+1}^i \Phi_k; M_T^i = Q_T^i; i \in \{2 \dots n\}$$

$$(4.57) \quad m_k^i = A_{k+1}' [M_{k+1}^i \phi_k + m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^i]; m_T^i = 0; i \in \{2 \dots n\}$$

$$(4.58) \quad L_k = Q_k^1 + A_{k+1}' L_{k+1} \Phi_k + \sum_{j \in 2 \dots n} A_{k+1}' Q_{k+1}^j A_{k+1} C_k^j \\ + \sum_{j \in 2 \dots n} A_{k+1}' Q_{k+1}^j B_{k+1}^j (N_k^{1j} \Phi_k + N_k^{0j}); L_T = Q_T^1$$

$$(4.59) \quad l_k = A_{k+1}' (L_{k+1} \phi_k + l_{k+1}) - A_{k+1}' Q_{k+1}^1 \tilde{x}_{k+1}^1 + \sum_{j \in 2 \dots n} A_{k+1}' Q_{k+1}^j A_{k+1} c_k^j \\ + \sum_{j \in 2 \dots n} A_{k+1}' Q_{k+1}^j B_{k+1}^j (N_k^{1j} \phi_k + n_k^j); l_T = 0$$

$$(4.60) \quad C_{k+1}^i = A_{k+1} C_k^i \Phi_k^{-1} + B_{k+1}^i (N_k^{1i} + N_k^{0i} \Phi_k^{-1}); C_0^i = 0; i \in \{2 \dots n\}$$

$$(4.61) \quad c_{k+1}^i = A_{k+1}(-C_k^i \Phi_k^{-1} \phi_k + c_k^i) + B_{k+1}^i(-N_k^{0i} \Phi_k^{-1} \phi_k + n_k^i); c_0^i = 0; i \in \{2 \dots n\}$$

$$(4.62) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = P_{k+1}^i x_k^* + \alpha_{k+1}^i$$

$$(4.63) \quad P_{k+1}^1 = W_{k+1}^1 \Phi_k + W_{k+1}^0$$

$$(4.64) \quad \alpha_{k+1}^1 = W_{k+1}^1 \phi_k + w_{k+1}$$

$$(4.65) \quad P_{k+1}^i = T_{k+1}^i \Phi_k; i \in \{2 \dots n\}$$

$$(4.66) \quad \alpha_{k+1}^i = T_{k+1}^i \phi_k + t_{k+1}^i; i \in \{2 \dots n\}$$

PROOF:

Theorem (8) can be applied to the given affine-quadratic game, since all conditions are satisfied for the given state equation (4.42) and cost functionals (4.43). Furthermore g_k^i is strictly convex in u_k^i . This can be seen by applying Corollary (1) to (4.67). Therefore there has to be a unique optimal equilibrium solution.

$$(4.67) \quad \frac{\partial^2}{\partial u_k^{i2}} g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = B_k^{i'} Q_k^i B_k^i + R_k^{ii}$$

To obtain relations which satisfy this unique solution we have to adapt (4.32) - (4.41) to the given state equation and cost functionals. This yields

$$\begin{aligned}
 (4.68) \quad H_k^1 = & \frac{1}{2}(x_k' Q_k^1 x_k + \sum_{j \in N} u_k^{j'} R_k^{1j} u_k^j) + \frac{1}{2}(\tilde{x}_k^{1'} Q_k^1 \tilde{x}_k^1 + \sum_{j \in N} \tilde{u}_k^{1j'} R_k^{1j} \tilde{u}_k^{1j}) \\
 & - \tilde{x}_k^{1'} Q_k^1 x_k - \sum_{j \in N} \tilde{u}_k^{1j'} R_k^{1j} u_k^j + \lambda_k'(A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k) \\
 & + \sum_{j \in 2 \dots n} \mu_{k-1}^{j'} A_k' [p_k^j + Q_k^j (x_k^* - \tilde{x}_k^j)] \\
 & + \sum_{j \in 2 \dots n} v_{k-1}^{j'} (B_k^{j'} Q_k^j x_k + R_k^{jj} u_k^j - B_k^{j'} Q_k^j \tilde{x}_k^{j'} - R_k^{jj} \tilde{u}_k^{jj} + B_k^{j'} p_k^j)
 \end{aligned}$$

$$\begin{aligned}
 (4.69) \quad H_k^i = & \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) \\
 & - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j + p_k^{i'} (A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k); i \in \{2 \dots n\}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial}{\partial u_k^1} (4.68) = 0 \Rightarrow & B_k^{1'} Q_k^1 (x_k^* - \tilde{x}_k^1) + R_k^{11} (u_k^{1*} - \tilde{u}_k^{11}) + B_k^{1'} \lambda_k \\
 & + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j A_k \mu_{k-1}^j + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j B_k^j v_{k-1}^j = 0
 \end{aligned}$$

$$\begin{aligned}
 (4.70) \quad u_k^{1*} = & -(R_k^{11})^{-1} (B_k^{1'} Q_k^1 (x_k^* - \tilde{x}_k^1) + B_k^{1'} \lambda_k \\
 & + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j A_k \mu_{k-1}^j + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j B_k^j v_{k-1}^j) + \tilde{u}_k^{11}
 \end{aligned}$$

$$\begin{aligned}
 (4.71) \quad \frac{\partial}{\partial u_k^i} (4.68) = 0 \Rightarrow & B_k^{i'} Q_k^1 (x_k^* - \tilde{x}_k^1) + R_k^{1i} (u_k^{1*} - \tilde{u}_k^{11}) + B_k^{i'} \lambda_k \\
 & + \sum_{j \in 2 \dots n} B_k^{i'} Q_k^j A_k \mu_{k-1}^j + (B_k^{i'} Q_k^i B_k^i + R_k^{ii}) v_{k-1}^i + \sum_{j \in 2 \dots n, j \neq i} B_k^{i'} Q_k^j B_k^j v_{k-1}^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

$$(4.72) \quad \lambda_{k-1} = A'_k Q_k^1 (x_k^* - \tilde{x}_k^1) + A'_k \lambda_k \\ + \sum_{j \in 2 \dots n} A'_k Q_k^j A_k \mu_{k-1}^j + \sum_{j \in 2 \dots n} A'_k Q_k^j B_k^j v_{k-1}^j; \lambda_T = 0$$

$$(4.73) \quad \mu_k^i = A_k \mu_{k-1}^i + B_k^i v_{k-1}^i; \mu_0^i = 0; i \in \{2 \dots n\}$$

$$\frac{\partial}{\partial u_i}(4.69) = 0 \Rightarrow B_k^{i'} Q_k^i x_k^* + R_k^{ii} u_k^{i*} - B_k^{i'} Q_k^i \tilde{x}_k^{i'} - R_k^{ii} \tilde{u}_k^{ii} + B_k^{i'} p_k^i = 0; i \in \{2 \dots n\}$$

$$(4.74) \quad u_k^{i*} = -(R_k^{ii})^{-1} B_k^{i'} (Q_k^i (x_k^* - \tilde{x}_k^{i'}) + p_k^i) + \tilde{u}_k^{ii}; i \in \{2 \dots n\}$$

$$(4.75) \quad p_{k-1}^{i*} = A'_k [p_k^i + Q_k^i (x_k^* - \tilde{x}_k^i)]; p_T^i = 0; i \in \{2 \dots n\}$$

$$(4.76) \quad x_k^* = A_k x_{k-1}^* + \sum_{j \in N} B_k^j u_k^{j*} + s_k; x_0^* = x_0$$

In the following induction arguments we will give proof that (4.77) - (4.79) are valid and the recursive relations for C_{k+1}^i , c_{k+1}^i , M_k^i , m_k^i , L_k and l_k (stated in the above theorem) are correct.

$$(4.77) \quad p_k^i = (M_k^i - Q_k^i) x_k^* + m_k^i; i \in \{2 \dots n\}$$

$$(4.78) \quad \lambda_k = (L_k - Q_k^1) x_k^* + l_k$$

$$(4.79) \quad \mu_k^i = C_k x_k^* + c_k ; i \in \{2 \dots n\}$$

Basis:

The induction for p_k^i and λ_k starts at $k = T$. First we make use of the general optimality conditions at stage T .

$$(4.75)_{k=T} \quad p_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.72)_{k=T} \quad \lambda_T = 0$$

Now we can substitute p_T^i and λ_T with functions affinely dependent on x_T .

$$(4.77)_{k=T} \quad p_T^i = (M_T^i - Q_T^i)x_T^* + m_T^i ; i \in \{2 \dots n\}$$

$$(M_T^i - Q_T^i)x_T^* + m_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.78)_{k=T} \quad \lambda_T = (L_T - Q_T^1)x_T^* + l_T$$

$$(L_T - Q_T^1)x_T^* + l_T = 0 ; i \in \{2 \dots n\}$$

Comparing coefficients gives

$$(4.56)_{k=T} \quad M_T^{ix} = Q_T^i ; i \in \{2 \dots n\}$$

$$(4.57)_{k=T} \quad m_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.58)_{k=T} \quad L_T = Q_T^1$$

$$(4.59)_{k=T} \quad l_T = 0$$

The induction for μ_k^i starts at $k = 0$. First we make use of the general optimality conditions at stage 0.

$$(4.73)_{k=0} \quad \mu_0^i = 0 ; i \in \{2 \dots n\}$$

Now we can substitute the μ_k^i with functions linearly dependent on x_0 .

$$(4.79)_{k=0} \quad \mu_0^i = C_0^i x_0^* + c_0^i ; i \in \{2 \dots n\}$$

$$C_0^i x_0^* + c_0^i = 0 ; i \in \{2 \dots n\}$$

Comparing coefficients gives

$$(4.60)_{k=0} \quad C_0^i = 0 ; i \in \{2 \dots n\}$$

$$(4.61)_{k=0} \quad c_0^i = 0 ; i \in \{2 \dots n\}$$

The interwoven inductive steps:

To prove the statements given in the above theorem, two inductive steps are combined.

As an induction hypothesis, the system of equations (4.77) and equation (4.78) are assumed to be true at stage $l+2$ and the system of equations (4.79) is assumed to be true at stage l . Now we have to show that these equations are fulfilled at stage $l+1$ and determine the corresponding recursive relations for C_{l+1}^i , c_{l+1}^i , M_l^i , m_l^i , L_l and l_l .

$$(4.77)_{k=l+2} \quad p_{l+1}^i = (M_{l+1}^i - Q_{l+1}^i)x_{l+1}^* + m_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.78)_{k=l+2} \quad \lambda_{l+1} = (L_{l+1} - Q_{l+1}^1)x_{l+1}^* + l_{l+1}$$

$$(4.79)_{k=l} \quad \mu_l^i = C_l^i x_l^* + c_l^i ; i \in \{2 \dots n\}$$

First these induction hypotheses are used in the general optimality conditions for p_k^i , λ_k and μ_k^i at stage $l+1$.

$$(4.75)_{k=l+1} \quad p_l^{i*} = A'_{l+1}[p_{l+1}^i + Q_{l+1}^i(x_{l+1}^* - \tilde{x}_{l+1}^i)] ; i \in \{2 \dots n\}$$

$$p_l^{i*} = A'_{l+1}[M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i] ; i \in \{2 \dots n\}$$

$$(4.72)_{k=l+1} \quad \lambda_l = A'_{l+1}Q_{l+1}^1(x_{l+1}^* - \tilde{x}_{l+1}^1) + A'_{l+1}\lambda_{l+1} \\ + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j v_l^j$$

$$\begin{aligned} \lambda_l = & A'_{l+1}(L_{l+1}x_{l+1}^* + l_{l+1}) - A'_{l+1}Q_{l+1}^1\tilde{x}_{l+1}^1 \\ & + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j v_l^j \end{aligned}$$

$$(4.73)_{k=l+1} \quad \mu_{l+1}^i = A_{l+1} \mu_l^i + B_{l+1}^i v_l^i$$

$$\mu_{l+1}^i = A_{l+1}(C_l^i x_l^* + c_l^i) + B_{l+1}^i v_l^i$$

To complete the inductive step we have to prove that the p_l^i and λ_l can be written as affine functions of x_l and that the μ_{l+1}^i can be written as affine functions of x_{l+1} . Therefore an interrelation between x_l and x_{l+1} which does not depend on the controls of the players nor on costate $(p_{l+1}^i, \lambda_{l+1}^i, \mu_l^i)$ or cocontrol (v_l^i) variables has to be deduced. To do so, first we have to substitute $u_{l+1}^{1*}, \dots, u_{l+1}^{n*}$ in the equation stated below for the evolution of the optimal state vector x_{l+1}^* by terms which are affine in x_l and x_{l+1} and furthermore only contain $C_l^i, c_l^i, M_{l+1}^i, m_{l+1}^i, L_{l+1}, l_{l+1}$ and matrices and vectors given by the game definition.

$$(4.76)_{k=l+1} \quad x_{l+1}^* = A_{l+1}x_l^* + \sum_{j \in N} B_{l+1}^j u_{l+1}^{j*} + s_{l+1}$$

In the first instance the optimality condition for u_{l+1}^{i*} (defined for $i \in \{2 \dots n\}$) can be rewritten with the help of $(4.77)_{k=l+1}$, which are the induction hypotheses for p_{l+1}^i .

$$(4.74)_{k=l+1} \quad u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (Q_{l+1}^i (x_{l+1}^* - \tilde{x}_{l+1}^{i'}) + p_{l+1}^i) + \tilde{u}_{l+1}^{ii}; i \in \{2 \dots n\}$$

$$u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$(4.80) \quad u_{l+1}^{i*} = T_{l+1}^i x_{l+1}^* + t_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.81) \quad T_{l+1}^i x_{l+1}^* + t_{l+1}^i = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} ; i \in \{2 \dots n\}$$

By comparing coefficients it follows that

$$(4.81)_{x_{l+1}^*} = (4.51)_{k=l} \quad T_{l+1}^i = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.81)_{const.} = (4.52)_{k=l} \quad t_{l+1}^i = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} ; i \in \{2 \dots n\}$$

The optimality condition for u_{l+1}^{1*} can be rewritten with the help of $(4.78)_{k=l+1}$ and $(4.79)_{k=l}$, which are the induction hypotheses for λ_{l+1} and μ_l^i .

$$(4.70)_{k=l+1} \quad u_{l+1}^{1*} = -(R_{l+1}^{11})^{-1} (B_{l+1}^{1'} Q_{l+1}^1 (x_{l+1}^* - \tilde{x}_{l+1}^1) + B_{l+1}^{1'} \lambda_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j v_l^j) + \tilde{u}_{l+1}^{11}$$

$$(4.82) \quad u_{l+1}^{1*} = -(R_{l+1}^{11})^{-1} (-B_{l+1}^{1'} Q_{l+1}^1 \tilde{x}_{l+1}^1 + B_{l+1}^{1'} (L_{l+1} x_{l+1}^* + l_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} (C_l^j x_l^* + c_l^j) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j v_l^j) + \tilde{u}_{l+1}^{11}$$

To make u_{l+1}^{1*} only dependent on C_l^i , c_l^i , M_{l+1}^i , m_{l+1}^i , L_{l+1} , l_{l+1} and matrices and vectors given by the game definition, we also have to substitute v_l^j ($j \in \{2 \dots n\}$) by a term affine in x_l and x_{l+1} . For that purpose v_l^j has to be explicated from $(4.71)_{l+1}$, because this is the only optimality condition at stage $l+1$ which has not been used

sofar. As a start λ_{l+1} , μ_l^i and u_{l+1}^{i*} ($i \in \{2 \dots n\}$) are substituted with the help of (4.78) _{$k=l+1$} , (4.79) _{$k=l$} and (4.80).

$$\begin{aligned}
 (4.71)_{k=l+1} \quad & B_{l+1}^{i'} Q_{l+1}^1 (x_{l+1}^* - \tilde{x}_{l+1}^{i*}) + R_{l+1}^{1i} (u_{l+1}^{i*} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} \lambda_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j \\
 & + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) v_l^i + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j v_l^j = 0; i \in \{2 \dots n\} \\
 \\
 & - B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^{i*} + R_{l+1}^{1i} (-(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) \\
 & + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} (L_{l+1} x_{l+1}^* + l_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} (C_l^j x_l^* + c_l^j) \\
 & + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) v_l^i + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j v_l^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

The above equations only contain constant expressions and terms linear in x_l or x_{l+1} . This fact justifies the following substitutions.

$$(4.83) \quad v_l^i = N_l^{1i} x_{l+1}^* + N_l^{0i} x_l^* + n_l^i; i \in \{2 \dots n\}$$

$$\begin{aligned}
 (4.84) \quad & - B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^{i*} + R_{l+1}^{1i} (-(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) \\
 & + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} (L_{l+1} x_{l+1}^* + l_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} (C_l^j x_l^* + c_l^j) \\
 & + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) (N_l^{1i} x_{l+1}^* + N_l^{0i} x_l^* + n_l^i) \\
 & + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j (N_l^{1j} x_{l+1}^* + N_l^{0j} x_l^* + n_l^j) = 0; i \in \{2 \dots n\}
 \end{aligned}$$

Comparing coefficients gives the following three systems of equations that admit

unique solutions N_l^{1i} , N_l^{0i} and n_l^i ($i \in \{2, \dots, n\}$) per assumption.

$$(4.84)_{x_{l+1}^*} = (4.53)_{k=l} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^i + B_{l+1}^{i'} L_{l+1} + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) N_l^{1i} + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j N_l^{1j} = 0; i \in \{2 \dots n\}$$

$$(4.84)_{x_l^*} = (4.54)_{k=l} \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} C_l^j + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) N_l^{0i} + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j N_l^{0j} = 0; i \in \{2 \dots n\}$$

$$(4.84)_{const.} = (4.55)_{k=l} - B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^{i*} + R_{l+1}^{1i} (- (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} l_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} c_l^j + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) n_l^i + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j n_l^j = 0; i \in \{2 \dots n\}$$

Using the elaborated relation for v_l^i ($i \in \{2 \dots n\}$) in (4.82) yields

$$u_{l+1}^{1*} = - (R_{l+1}^{11})^{-1} (-B_{l+1}^{1'} Q_{l+1}^1 \tilde{x}_{l+1}^1 + B_{l+1}^{1'} (L_{l+1} x_{l+1}^* + l_{l+1})) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} (C_l^j x_l^* + c_l^j) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j (N_l^{1j} x_{l+1}^* + N_l^{0j} x_l^* + n_l^j) + \tilde{u}_{l+1}^{11}$$

The structure of the above equation justifies the following substitution

$$(4.85) \quad u_{l+1}^{1*} = W_{l+1}^1 x_{l+1}^* + W_{l+1}^0 x_l^* + w_{l+1}$$

$$\begin{aligned}
 (4.86) \quad W_{l+1}^1 x_{l+1}^* + W_{l+1}^0 x_l^* + w_{l+1} = & -(R_{l+1}^{11})^{-1} (-B_{l+1}^{1'} Q_{l+1}^1 \tilde{x}_{l+1}^j \\
 & + B_{l+1}^{1'} (L_{l+1} x_{l+1}^* + l_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} (C_l^j x_l^* + c_l^j) \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j (N_l^{1j} x_{l+1}^* + N_l^{0j} x_l^* + n_l^j)) + \tilde{u}_{l+1}^{11}
 \end{aligned}$$

By comparing coefficients it follows that

$$(4.86)_{x_{l+1}^*} = (4.48)_{k=l} \quad W_{l+1}^1 = -(R_{l+1}^{11})^{-1} (B_{l+1}^{1'} L_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j N_l^{1j})$$

$$(4.86)_{x_l^*} = (4.49)_{k=l} \quad W_{l+1}^0 = -(R_{l+1}^{11})^{-1} \left(\sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} C_l^j + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j N_l^{0j} \right)$$

$$\begin{aligned}
 (4.86)_{const.} = (4.50)_{k=l} \quad w_{l+1} = & -(R_{l+1}^{11})^{-1} (-B_{l+1}^{1'} Q_{l+1}^1 \tilde{x}_{l+1}^j + B_{l+1}^{1'} l_{l+1} \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} c_l^j + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j n_l^j) + \tilde{u}_{l+1}^{11}
 \end{aligned}$$

Now the control variables can be replaced in the optimal state equation by terms affine in x_l and x_{l+1} .

$$(4.76)_{k=l+1} \quad x_{l+1}^* = A_{l+1} x_l^* + B_{l+1}^1 u_{l+1}^{1*} + \sum_{j \in \{2 \dots n\}} B_{l+1}^j u_{l+1}^{j*} + s_{l+1}$$

$$x_{l+1}^* = A_{l+1} x_l^* + B_{l+1}^1 (W_{l+1}^1 x_{l+1}^* + W_{l+1}^0 x_l^* + w_{l+1}) + \sum_{j \in \{2 \dots n\}} B_{l+1}^j (T_{l+1}^j x_{l+1}^* + t_{l+1}^j) + s_{l+1}$$

Making x_{l+1} explicit gives

$$(I - B_{l+1}^1 W_{l+1}^1 - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^j) x_{l+1}^* = A_{l+1} x_l^* + B_{l+1}^1 (W_{l+1}^0 x_l^* + w_{l+1}) + \sum_{j \in \{2 \dots n\}} B_{l+1}^j t_{l+1}^j + s_{l+1}$$

$$x_{l+1}^* = (I - B_{l+1}^1 W_{l+1}^1 - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^j)^{-1} (A_{l+1} x_l^* + B_{l+1}^1 (W_{l+1}^0 x_l^* + w_{l+1}) + \sum_{j \in \{2 \dots n\}} B_{l+1}^j t_{l+1}^j + s_{l+1})$$

The structure of the above equation justifies the following substitution

$$(4.45)_{k=l} \quad x_{l+1}^* = \Phi_l x_l^* + \phi_l$$

$$(4.87) \quad (I - B_{l+1}^1 W_{l+1}^1 - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^j) (\Phi_l x_l^* + \phi_l) = A_{l+1} x_l^* + B_{l+1}^1 (W_{l+1}^0 x_l^* + w_{l+1}) + \sum_{j \in \{2 \dots n\}} B_{l+1}^j t_{l+1}^j + s_{l+1}$$

Comparing coefficients gives

$$(4.87)_{x_l^*} = (4.46)_{k=l} \quad \Phi_l = (I - B_{l+1}^1 W_{l+1}^1 - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^j)^{-1} (A_{l+1} + B_{l+1}^1 W_{l+1}^0)$$

$$(4.87)_{const.} = (4.47)_{k=l} \quad \phi_l = (I - B_{l+1}^1 W_{l+1}^1 - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^j)^{-1} (B_{l+1}^1 w_{l+1} + \sum_{j \in \{2 \dots n\}} B_{l+1}^j t_{l+1}^j + s_{l+1})$$

The above relation between x_{l+1} and x_l can be used to finish the inductive step of p_l^i , λ_l and μ_{l+1}^i . As a start $(4.45)_{k=l}$ is used to continue the derivation of p_l^i .

$$p_l^{i*} = A'_{l+1}[M_{l+1}^i x_{l+1}^* + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i] ; i \in \{2 \dots n\}$$

$$p_l^{i*} = A'_{l+1}[M_{l+1}^i (\Phi_l x_l^* + \phi_l) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i] ; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$(4.77)_{k=l+1} \quad p_l^i = (M_l^i - Q_l^2) x_l^* + m_l^i ; i \in \{2 \dots n\}$$

$$(4.88) \quad (M_l^i - Q_l^2) x_l^* + m_l^i = A'_{l+1}[M_{l+1}^i (\Phi_l x_l^* + \phi_l) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i] ; i \in \{2 \dots n\}$$

By comparing coefficients it follows that

$$(4.88)_{x_l^*} = (4.56)_{k=l} \quad M_l^i = Q_l^i + A'_{l+1} M_{l+1}^i \Phi_l ; i \in \{2 \dots n\}$$

$$(4.88)_{const.} = (4.57)_{k=l} \quad m_l^i = A'_{l+1}[M_{l+1}^i \phi_l + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i] ; i \in \{2 \dots n\}$$

As a next step $(4.45)_{k=l}$, $(4.73)_{k=l+1}$ and (4.83) are used to continue the derivation of λ_l .

$$\begin{aligned} \lambda_l = & A'_{l+1}(L_{l+1} x_{l+1}^* + l_{l+1}) - A'_{l+1} Q_{l+1}^1 \tilde{x}_{l+1}^1 \\ & + \sum_{j \in 2 \dots n} A'_{l+1} Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} A'_{l+1} Q_{l+1}^j B_{l+1}^j v_l^j \end{aligned}$$

$$\begin{aligned}\lambda_l &= A'_{l+1}(L_{l+1}x_{l+1}^* + l_{l+1}) - A'_{l+1}Q_{l+1}^1\tilde{x}_{l+1}^1 \\ &+ \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}(C_l^j x_l^* + c_l^j) + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{1j} x_{l+1}^* + N_l^{0j} x_l^* + n_l^j)\end{aligned}$$

$$\begin{aligned}\lambda_l &= A'_{l+1}(L_{l+1}(\Phi_l x_l^* + \phi_l) + l_{l+1}) - A'_{l+1}Q_{l+1}^1\tilde{x}_{l+1}^1 + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}(C_l^j x_l^* + c_l^j) \\ &+ \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{1j}(\Phi_l x_l^* + \phi_l) + N_l^{0j} x_l^* + n_l^j)\end{aligned}$$

The structure of the above equation justifies the following substitution

$$(4.78)_{k=l+1} \quad \lambda_l = (L_l - Q_l^1)x_l^* + l_l$$

$$\begin{aligned}(4.89) \quad (L_l - Q_l^1)x_l^* + l_l &= A'_{l+1}(L_{l+1}(\Phi_l x_l^* + \phi_l) + l_{l+1}) - A'_{l+1}Q_{l+1}^1\tilde{x}_{l+1}^1 \\ &+ \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}(C_l^j x_l^* + c_l^j) + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{1j}(\Phi_l x_l^* + \phi_l) + N_l^{0j} x_l^* + n_l^j)\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}(4.89)_{x_l^*} &= (4.58)_{k=l} \quad L_l = Q_l^1 + A'_{l+1}L_{l+1}\Phi_l + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}C_l^j \\ &+ \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{1j}\Phi_l + N_l^{0j})\end{aligned}$$

$$\begin{aligned}(4.89)_{const.} &= (4.59)_{k=l} \quad l_l = A'_{l+1}(L_{l+1}\phi_l + l_{l+1}) - A'_{l+1}Q_{l+1}^1\tilde{x}_{l+1}^1 \\ &+ \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}c_l^j + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{1j}\phi_l + n_l^j)\end{aligned}$$

Eventually (4.45)_{k=l} and (4.83) are utilized to deduce a relation for μ_{l+1} that is affinely dependent on x_{l+1} .

$$\mu_{l+1}^i = A_{l+1}(C_l^i x_l^* + c_l^i) + B_{l+1}^i v_l^i$$

$$\mu_{l+1}^i = A_{l+1}(C_l^i x_l^* + c_l^i) + B_{l+1}^i (N_l^{1i} x_{l+1}^* + N_l^{0i} x_l^* + n_l^i)$$

$$\mu_{l+1}^i = A_{l+1}(C_l^i \Phi_l^{-1}(x_{l+1}^* - \phi_l) + c_l^i) + B_{l+1}^i (N_l^{1i} x_{l+1}^* + N_l^{0i} \Phi_l^{-1}(x_{l+1}^* - \phi_l) + n_l^i)$$

The structure of the above equations justifies the following substitutions

$$\mu_{l+1}^i = C_{l+1}^i x_{l+1}^* + c_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.90) \quad C_{l+1}^i x_{l+1}^* + c_{l+1}^i = A_{l+1}(C_l^i \Phi_l^{-1}(x_{l+1}^* - \phi_l) + c_l^i) + B_{l+1}^i (N_l^{1i} x_{l+1}^* + N_l^{0i} \Phi_l^{-1}(x_{l+1}^* - \phi_l) + n_l^i) ; i \in \{2 \dots n\}$$

By comparing coefficients it follows that

$$(4.90)_{x_{l+1}^*} = (4.60)_{k=l} \quad C_{l+1}^i = A_{l+1} C_l^i \Phi_l^{-1} + B_{l+1}^i (N_l^{1i} + N_l^{0i} \Phi_l^{-1}) ; i \in \{2 \dots n\}$$

$$(4.90)_{const} = (4.61)_{k=l} \quad c_{l+1}^i = A_{l+1} (-C_l^i \Phi_l^{-1} \phi_l + c_l^i) + B_{l+1}^i (-N_l^{0i} \Phi_l^{-1} \phi_l + n_l^i) ; i \in \{2 \dots n\}$$

At this point the inductive step and hence the induction argument is completed, but we will try to transform $u_{l+1}^{1*}, \dots, u_{l+1}^{n*}$ so that their evolution depends affinely on x_l^* , the optimal equilibrium value of the state vector of the previous stage.

Let us start with u_{l+1}^{1*} , where we apply (4.45)_{k=l} to (4.85)

$$(4.85) \quad u_{l+1}^{1*} = W_{l+1}^1 x_{l+1}^* + W_{l+1}^0 x_l^* + w_{l+1}$$

$$u_{l+1}^{1*} = W_{l+1}^1 (\Phi_l x_l^* + \phi_l) + W_{l+1}^0 x_l^* + w_{l+1}$$

The structure of the above equation justifies the following substitution

$$u_{l+1}^{1*} = P_{l+1}^1 x_l + \alpha_{l+1}^1$$

$$(4.91) \quad P_{l+1}^1 x_l + \alpha_{l+1}^1 = W_{l+1}^1 (\Phi_l x_l^* + \phi_l) + W_{l+1}^0 x_l^* + w_{l+1}$$

Comparing coefficients gives

$$(4.91)_{x_l^*} = (4.63)_{k=l} \quad P_{l+1}^1 = W_{l+1}^1 \Phi_l + W_{l+1}^0$$

$$(4.91)_{const.} = (4.64)_{k=l} \quad \alpha_{l+1}^1 = W_{l+1}^1 \phi_l + w_{l+1}$$

Finally (4.45)_{k=l} is used in (4.80):

$$(4.80) \quad u_{l+1}^{i*} = T_{l+1}^i x_{l+1}^* + t_{l+1}^i ; i \in \{2 \dots n\}$$

$$u_{l+1}^{i*} = T_{l+1}^i (\Phi_l x_l^* + \phi_l) + t_{l+1}^i ; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$u_{l+1}^{i*} = P_{l+1}^i x_l + \alpha_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.92) \quad P_{l+1}^i x_l + \alpha_{l+1}^i = T_{l+1}^i (\Phi_l x_l^* + \phi_l) + t_{l+1}^i ; i \in \{2 \dots n\}$$

By comparing coefficients it follows that

$$(4.92)_{x_l^*} = (4.65)_{k=l} \quad P_{l+1}^i = T_{l+1}^i \Phi_l ; i \in \{2 \dots n\}$$

$$(4.92)_{const.} = (4.66)_{k=l} \quad \alpha_{l+1}^i = T_{l+1}^i \phi_l + t_{l+1}^i ; i \in \{2 \dots n\} \quad \square$$

4.2.3 Special case: "Interwoven-inductions-results" for linear-quadratic games with one leader and one follower

In this subsection we first specialize the results of the previous subsection (4.2.2) to a linear-quadratic 2-person game in Corollary (5) and then in Proposition (6) the specialized results are transformed into the terminology used in Corollary 7.1 in Başar and Olsder (1999, pp. 371-372)[2] to point out some serious mistakes stated there.

Corollary 5 *A 2-person linear-quadratic dynamic game (cf. Def. (4)) admits a unique open-loop Stackelberg equilibrium solution with one leader and one follower if*

- $Q_k^i \geq 0$ and $R_k^{ii} > 0$ (defined for $k \in K$, $i \in N$).
- $(I - B_{k+1}^1 W_{k+1}^1 - B_{k+1}^2 T_{k+1}^2)^{-1}, (A_{k+1} + B_{k+1}^1 W_{k+1}^0)^{-1}$ (defined for $k \in K$) exists.

If these conditions are satisfied, the unique equilibrium strategies are given by (4.106), where the associated state trajectory x_{k+1}^ is given by (4.96).²¹*

$$(4.93) \quad f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + B_k^1 u_k^1 + B_k^2 u_k^2; k \in K$$

$$(4.94) \quad L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, u_k^2, x_{k-1})$$

$$(4.95) \quad g_k^i(x_k, u_k^1, u_k^2, x_{k-1}) = \frac{1}{2}(x_k' Q_k^i x_k + u_k^{i'} u_k^i + u_k^{j'} R_k^{ij} u_k^j); k \in K; i, j \in \{1, 2\}; i \neq j$$

$$(4.96) \quad x_{k+1}^* = \Phi_k x_k^*; x_0^* = x_0$$

$$(4.97) \quad \Phi_k = (I - B_{k+1}^1 W_{k+1}^1 - B_{k+1}^2 T_{k+1}^2)^{-1} (A_{k+1} + B_{k+1}^1 W_{k+1}^0)$$

²¹ For all equations belonging to this corollary and its proof, $k \in \{0, \dots, T-1\}$ if nothing different is stated.

$$(4.98) \quad W_{k+1}^1 = -B_{k+1}^{1'} L_{k+1} - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^{12}$$

$$(4.99) \quad W_{k+1}^0 = -B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} C_k^2 - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^{02}$$

$$(4.100) \quad T_{k+1}^2 = -B_{k+1}^{2'} M_{k+1}^2$$

$$(4.101) \quad N_k^{12} = -(B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} L_{k+1} - R_{k+1}^{12} B_{k+1}^{2'} M_{k+1}^2)$$

$$(4.102) \quad N_k^{02} = -(B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} Q_{k+1}^2 A_{k+1} C_k^2)$$

$$(4.103) \quad M_k^2 = Q_k^2 + A_{k+1}' M_{k+1}^2 \Phi_k; M_T^2 = Q_T^2$$

$$(4.104) \quad L_k = Q_k^1 + A_{k+1}' (L_{k+1} \Phi_k + Q_{k+1}^2 A_{k+1} C_k^2 + Q_{k+1}^2 B_{k+1}^2 (N_k^{12} \Phi_k + N_k^{02})) ; L_T = Q_T^1$$

$$(4.105) \quad C_{k+1}^2 = A_{k+1} C_k^2 \Phi_k^{-1} + B_{k+1}^2 (N_k^{12} + N_k^{02} \Phi_k^{-1}) ; C_0 = 0$$

$$(4.106) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = P_{k+1}^i x_k^* ; i \in \{1, 2\}$$

$$(4.107) \quad P_{k+1}^1 = W_{k+1}^1 \Phi_k + W_{k+1}^0$$

$$(4.108) \quad P_{k+1}^2 = T_{k+1}^2 \Phi_k ; i \in \{2 \dots n\}$$

PROOF:

Corollary (5) is proven in the same way as Theorem (9) taking into consideration simplifications resulting from the different number of followers and the modified state equation and cost functionals. \square

Remark 15 *Special attention should be paid to the fact that the assumption about the existence of unique solutions of the systems of equations (4.53), (4.54) and (4.55) in Theorem (9) is equivalent to the existence of $(B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1}$ in this special case. But the existence of $(B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1}$ is assured because of the assumption made on Q_{k+1}^2 .*

Proposition 6 *The systems of equations defining the unique equilibrium strategies $\gamma_{k+1}^{i*}(x_0)$ ($i \in \{1, 2\}$) and the associated state trajectory x_{k+1}^* in Corollary (5) can also be written in the following way:²²*

$$(4.109) \quad x_{k+1}^* = \Phi_k x_k^*; x_0^* = x_0$$

$$(4.110) \quad \Phi_k = (I + B_{k+1}^1 B_{k+1}^{1'} (I + Q_{k+1}^2 B_{k+1}^2 B_{k+1}^{2'})^{-1} \Lambda_{k+1} + B_{k+1}^1 B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} R_{k+1}^{12} B_{k+1}^{2'} P_{k+1} + B_{k+1}^2 B_{k+1}^{2'} P_{k+1})^{-1} (A_{k+1} - B_{k+1}^1 B_{k+1}^{1'} Q_{k+1}^2 (I + B_{k+1}^2 B_{k+1}^{2'} Q_{k+1}^2)^{-1} A_{k+1} M_k)$$

$$(4.111) \quad P_k = Q_k^2 + A_{k+1}' P_{k+1} \Phi_k; P_T = Q_T^2$$

$$(4.112) \quad \Lambda_k = Q_k^1 + A_{k+1}' (\Lambda_{k+1} \Phi_k + Q_{k+1}^2 A_{k+1} M_k - Q_{k+1}^2 B_{k+1}^2 N_k); \Lambda_T = Q_T^1$$

$$(4.113) \quad M_{k+1} = (A_{k+1} M_k - B_k^2 N_k) \Phi_k^{-1}; M_0 = 0$$

²² For all equations belonging to this proposition and its proof, $k \in \{0, \dots, T-1\}$ if nothing different is stated. (4.111), (4.112) and (4.113) are wrong in Başar and Olsder.

$$(4.114) \quad N_{k+1} = (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} \Lambda_{k+1} \Phi_k - R_{k+1}^{12} B_{k+1}^{2'} K_{k+1}^2 + B_{k+1}^{2'} Q_{k+1}^2 A_{k+1} M_k)$$

$$(4.115) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = -B_{k+1}^{i'} K_{k+1}^i x_k^*; i \in \{1, 2\}$$

$$(4.116) \quad K_{k+1}^1 = [I + Q_k^2 B_k^2 B_k^{2'}]^{-1} \Lambda_{k+1} \Phi_k + Q_k^2 B_k^2 [I + B_k^{2'} Q_k^2 B_k^2]^{-1} R_k^{12} B_k^{2'} P_{k+1} \Phi_k + Q_k^2 [I + B_k^2 B_k^{2'} Q_k^2]^{-1} A_k M_k$$

$$(4.117) \quad K_{k+1}^2 = P_{k+1} \Phi_k; i \in \{2 \dots n\}$$

PROOF:

The proof is carried out by renaming the costate matrices and then showing that the relations for the state and costate matrices and for the control vectors of Corollary (5) can be rewritten in the way stated above.

Let us start by renaming the costate matrices C_k^2 , L_k and M_k^2 .

$$(4.118) \quad C_k^2 \hat{=} M_k; L_k \hat{=} \Lambda_k; M_k^2 \hat{=} P_k; k \in \{0, \dots, T\}$$

Next we prove that the costate matrices fulfill (4.111), (4.113) and (4.112) respectively.

Taking consideration of the renaming (4.103) gives

$$(4.111) \quad P_k = Q_k^2 + A_{k+1}' P_{k+1}^2 \Phi_k$$

Now we show (using (4.118)) that there is a relation between the cocontrol matrices (4.101), (4.102) and (4.114):

$$\begin{aligned} N_k^{12} \Phi_k + N_k^{02} &= -(B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} \Lambda_{k+1} - R_{k+1}^{12} B_{k+1}^{2'} P_{k+1}) \Phi_k \\ &\quad - (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} Q_{k+1}^2 A_{k+1} M_k) \end{aligned}$$

Making use of (4.117) gives

$$\begin{aligned} -N_k^{12} \Phi_k - N_k^{02} &= (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} \\ &\quad (B_{k+1}^{2'} \Lambda_{k+1} \Phi_k - R_{k+1}^{12} B_{k+1}^{2'} K_{k+1}^2) + (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} Q_{k+1}^2 A_{k+1} M_k) = N_{k+1} \end{aligned}$$

Therefore the following relation is satisfied

$$(4.119) \quad - (4.101) \Phi_k - (4.102) = (4.114)$$

Applying (4.118) and (4.119) to (4.105) yields

$$M_{k+1} = A_{k+1} M_k \Phi_k^{-1} + B_{k+1}^2 (N_k^{12} + N_k^{02} \Phi_k^{-1})$$

$$M_{k+1} = (A_{k+1} M_k + B_{k+1}^2 (N_k^{12} \Phi_k + N_k^{02})) \Phi_k^{-1}$$

$$(4.113) \quad M_{k+1} = (A_{k+1} M_k - B_{k+1}^2 N_{k+1}) \Phi_k^{-1}$$

Making use of (4.118) and (4.119) in (4.104) gives

$$\Lambda_k = Q_k^1 + A'_{k+1}(\Lambda_{k+1}\Phi_k + Q_{k+1}^2 A_{k+1} M_k + Q_{k+1}^2 B_{k+1}^2 (N_k^{12}\Phi_k + N_k^{02}))$$

$$(4.112) \quad \Lambda_k = Q_k^1 + A'_{k+1}(\Lambda_{k+1}\Phi_k + Q_{k+1}^2 A_{k+1} M_k - Q_{k+1}^2 B_{k+1}^2 N_{k+1})$$

Next we show the correctness of the relation of Φ_k stated in (4.110).

First use (4.98), (4.99), (4.100) and (4.118) in (4.97):

$$(4.97) \quad \Phi_k = (I - B_{k+1}^1 W_{k+1}^1 - B_{k+1}^2 T_{k+1}^2)^{-1} (A_{k+1} + B_{k+1}^1 W_{k+1}^0)$$

$$\begin{aligned} \Phi_k = & (I + B_{k+1}^1 (B_{k+1}^{1'} \Lambda_{k+1} + B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^{12}) + B_{k+1}^2 B_{k+1}^{2'} P_{k+1})^{-1} \\ & (A_{k+1} - B_{k+1}^1 (B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} M_k + B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^{02})) \end{aligned}$$

Substituting N_k^{12} with the help of (4.101) yields

$$\begin{aligned} \Phi_k = & (I + B_{k+1}^1 (B_{k+1}^{1'} \Lambda_{k+1} - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} \\ & (B_{k+1}^{2'} \Lambda_{k+1} - R_{k+1}^{12} B_{k+1}^{2'} P_{k+1})) + B_{k+1}^2 B_{k+1}^{2'} P_{k+1})^{-1} (A_{k+1} - B_{k+1}^1 \\ & (B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} M_k - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} B_{k+1}^{2'} Q_{k+1}^2 A_{k+1} M_k)) \end{aligned}$$

Putting together Λ_{k+1} and M_k gives

$$\begin{aligned} \Phi_k = & (I + B_{k+1}^1 B_{k+1}^{1'} (I - Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} B_{k+1}^{2'}) \Lambda_{k+1} \\ & + B_{k+1}^1 B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} R_{k+1}^{12} B_{k+1}^{2'} P_{k+1} + B_{k+1}^2 B_{k+1}^{2'} P_{k+1})^{-1} \\ & (A_{k+1} - B_{k+1}^1 B_{k+1}^{1'} Q_{k+1}^2 (I - B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} B_{k+1}^{2'} Q_{k+1}^2) A_{k+1} M_k) \end{aligned}$$

Finally applying Lemma (4) 1. and 2. to the particular expressions above leads to

$$(4.110) \quad \Phi_k = (I + B_{k+1}^1 B_{k+1}^{1'} (I + Q_{k+1}^2 B_{k+1}^2 B_{k+1}^{2'})^{-1} \Lambda_{k+1} \\ + B_{k+1}^1 B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} R_{k+1}^{12} B_{k+1}^{2'} P_{k+1} \\ + B_{k+1}^2 B_{k+1}^{2'} P_{k+1})^{-1} (A_{k+1} - B_{k+1}^1 B_{k+1}^{1'} Q_{k+1}^2 (I + B_{k+1}^2 B_{k+1}^{2'} Q_{k+1}^2)^{-1} A_{k+1} M_k)$$

Eventually the correctness of the rewritten equilibrium strategies γ_{k+1}^{i*} ($i \in \{1, 2\}$, $k \in \{0, \dots, T\}$), given by (4.115) - (4.117), has to be shown.

First note that (4.106) $\hat{=}$ (4.115) if and only if $P_{k+1}^i = -B_{k+1}^i K_{k+1}^i$.

$$(4.106) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = P_{k+1}^i x_k^*$$

$$(4.115) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = -B_{k+1}^{i'} K_{k+1}^i x_k^*$$

Therefore, as a start use (4.98) and (4.99) in (4.107) (also considering (4.118))

$$(4.107) \quad P_{k+1}^1 = W_{k+1}^1 \Phi_k + W_{k+1}^0$$

$$P_{k+1}^1 = -(B_{k+1}^{1'} \Lambda_{k+1} + B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^{12}) \Phi_k - (B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} M_k + B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^{02})$$

Next substitute $N_k^{12} \Phi_k + N_k^{02}$ with the help of (4.119)

$$P_{k+1}^1 = -B_{k+1}^{1'} \Lambda_{k+1} \Phi_k - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (N_k^{12} \Phi_k + N_k^{02}) - B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} M_k$$

$$P_{k+1}^1 = -B_{k+1}^{1'} \Lambda_{k+1} \Phi_k + B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_{k+1} - B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} M_k$$

Now make use of (4.114)

$$P_{k+1}^1 = -B_{k+1}^{1'} \Lambda_{k+1} \Phi_k + B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 ((B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} \Lambda_{k+1} \Phi_k - R_{k+1}^{12} B_{k+1}^{2'} K_{k+1}^2) + (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} (B_{k+1}^{2'} Q_{k+1}^2 A_{k+1} M_k)) - B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} M_k$$

Putting together Λ_{k+1} and M_k gives

$$\begin{aligned} P_{k+1}^1 = & -B_{k+1}^{1'} (I - Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} B_{k+1}^{2'}) \Lambda_{k+1} \Phi_k \\ & - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} R_{k+1}^{12} B_{k+1}^{2'} P_{k+1} \Phi_k \\ & - B_{k+1}^{1'} Q_{k+1}^2 (I - B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} B_{k+1}^{2'} Q_{k+1}^2) A_{k+1} M_k \end{aligned}$$

Finally applying Lemma (4) 1. and 2. to the particular expressions above and using (4.116) leads to

$$\begin{aligned} P_{k+1}^1 = & -B_{k+1}^{1'} [(I + Q_{k+1}^2 B_{k+1}^2 B_{k+1}^{2'})^{-1} \Lambda_{k+1} \Phi_k + Q_{k+1}^2 B_{k+1}^2 (B_{k+1}^{2'} Q_{k+1}^2 B_{k+1}^2 + I)^{-1} \\ & R_{k+1}^{12} B_{k+1}^{2'} P_{k+1} \Phi_k + Q_{k+1}^2 (I + B_{k+1}^2 B_{k+1}^{2'} Q_{k+1}^2)^{-1} A_{k+1} M_k] \end{aligned}$$

$$(4.120) \quad P_{k+1}^1 = -B_{k+1}^{1'} K_{k+1}^1$$

As a last step $P_{k+1}^2 = -B_{k+1}^{2'} K_{k+1}^2$ can be shown by using (4.100) in (4.108) (also considering (4.118))

$$(4.121) \quad P_{k+1}^2 = T_{k+1}^2 \Phi_k$$

$$(4.122) \quad P_{k+1}^2 = -B_{k+1}^{2'} P_{k+1} \Phi_k = -B_{k+1}^{2'} K_{k+1}^2 \quad \square$$

Remark 16 *The equations given in Proposition (6) are the same as in Başar and Olsder (1999, p.371) [2] (except for the three equations already mentionned above) if k is replaced by k' , whereas $k'=k+1$ and $k' \in K$, and the indices of the state vector x and of the matrices related to it (M, P, Λ, Φ, Q^i) are augmented by 1.*

4.2.4 The one-induction-results for affine-quadratic games with one leader and arbitrarily many followers

In the following, the results of Theorem (8) are applied to an affine-quadratic dynamic game with one leader and arbitrarily many followers. Theorem (10) presents equilibrium equations that can easily be used for an algorithmic disintegration of the given Stackelberg game.

Theorem 10 *An n -person affine-quadratic dynamic game (cf. Def. (4)) admits a unique open-loop Stackelberg equilibrium solution with one leader and arbitrarily many followers if*

- $Q_k^i \geq 0, R_k^{ii} > 0$ (defined for $k \in K, i \in N$).
- $(I - B_{k+1}^1 W_{k+1}^x - \sum_{j \in 2 \dots n} B_{k+1}^j T_{k+1}^{jx})^{-1}$ (defined for $k \in K$) exists.
- (4.141), (4.142) and (4.143) admit unique solutions $N_k^{ix}, N_k^{ij\mu}$ and n_k^i (defined for: $k \in K, i, j \in \{2, \dots, n\}$).

If these conditions are satisfied, the unique equilibrium strategies are given by (4.150), where the associated state trajectory x_{k+1}^* is given by (4.126).²³

$$(4.123) \quad f_{k-1}(x_{k-1}, u_k^1, \dots, u_k^n) = A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k; k \in K$$

$$(4.124) \quad L^i(x_0, u^1, \dots, u^n) = \sum_{k=1}^T g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$$

$$(4.125) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j; k \in K$$

²³ For all equations belonging to this theorem and its proof, $i \in N$ and $k \in \{0, \dots, T-1\}$ if nothing different is stated.

$$(4.126) \quad x_{k+1}^* = \Phi_k^x x_k^* + \sum_{j \in 2 \dots n} \Phi_k^{j\mu} \mu_k^j + \phi_k; \quad x_0^* = x_0$$

$$(4.127) \quad \Phi_k^x = (I - B_{k+1}^1 W_{k+1}^x - \sum_{j \in 2 \dots n} B_{k+1}^j T_{k+1}^{jx})^{-1} A_{k+1}$$

$$(4.128) \quad \Phi_k^{i\mu} = (I - B_{k+1}^1 W_{k+1}^x - \sum_{j \in 2 \dots n} B_{k+1}^j T_{k+1}^{jx})^{-1} \\ (B_{k+1}^1 W_{k+1}^{i\mu} + \sum_{j \in 2 \dots n} B_{k+1}^j T_{k+1}^{ji\mu}); \quad i \in \{2 \dots n\}$$

$$(4.129) \quad \phi_k = (I - B_{k+1}^1 W_{k+1}^x - \sum_{j \in 2 \dots n} B_{k+1}^j T_{k+1}^{jx})^{-1} \\ (B_{k+1}^1 w_{k+1} + \sum_{j \in 2 \dots n} B_{k+1}^j t_{k+1}^j + s_{k+1})$$

$$(4.130) \quad \mu_{k+1}^i = \Psi_k^{ix} x_k^* + \sum_{j \in 2 \dots n} \Psi_k^{ij\mu} \mu_k^j + \psi_k^i; \quad \mu_0^i = 0; \quad i \in \{2 \dots n\}$$

$$(4.131) \quad \Psi_k^{ix} = B_{k+1}^i N_k^{ix} \Phi_k^x; \quad i \in \{2 \dots n\}$$

$$(4.132) \quad \Psi_k^{ii} = A_{k+1} + B_{k+1}^i (N_k^{ix} \Phi_k^{i\mu} + N_k^{ii\mu}); \quad i \in \{2 \dots n\}$$

$$(4.133) \quad \Psi_k^{im} = B_{k+1}^i (N_k^{ix} \Phi_k^{m\mu} + N_k^{im\mu}); \quad i, m \in \{2 \dots n\}; \quad m \neq i$$

$$(4.134) \quad \psi_k^i = B_{k+1}^i (N_k^{ix} \phi_k + n_k^i); i \in \{2 \dots n\}$$

$$(4.135) \quad W_{k+1}^x = -(R_{k+1}^{11})^{-1} (B_{k+1}^{1'} (L_{k+1}^x + \sum_{j \in 2 \dots n} L_{k+1}^{j\mu} B_{k+1}^j N_k^{jx}) \\ + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j B_{k+1}^j N_k^{jx})$$

$$(4.136) \quad W_{k+1}^{m\mu} = -(R_{k+1}^{11})^{-1} (B_{k+1}^{1'} (L_{k+1}^{m\mu} A_{k+1} + \sum_{j \in 2 \dots n} L_{k+1}^{j\mu} B_{k+1}^j N_k^{jm\mu}) \\ + B_{k+1}^{1'} Q_{k+1}^m A_{k+1} + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j B_{k+1}^j N_k^{jm\mu}); m \in \{2 \dots n\}$$

$$(4.137) \quad w_{k+1} = -(R_{k+1}^{11})^{-1} (-B_{k+1}^{1'} Q_{k+1}^1 \tilde{x}_{k+1}^1 + B_{k+1}^{1'} (\sum_{j \in 2 \dots n} L_{k+1}^{j\mu} B_{k+1}^j n_k^j \\ + l_{k+1}) + \sum_{j \in 2 \dots n} B_{k+1}^{1'} Q_{k+1}^j B_{k+1}^j n_k^j) + \tilde{u}_{k+1}^{11}$$

$$(4.138) \quad T_{k+1}^{ix} = -(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (M_{k+1}^{ix} + \sum_{j \in 2 \dots n} M_{k+1}^{ij\mu} B_{k+1}^j N_k^{jx}); i \in \{2 \dots n\}$$

$$(4.139) \quad T_{k+1}^{im\mu} = -(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (M_{k+1}^{im\mu} A_{k+1} + \\ \sum_{j \in 2 \dots n} M_{k+1}^{ij\mu} B_{k+1}^j N_k^{jm\mu}); i, m \in \{2 \dots n\}$$

$$(4.140) \quad t_{k+1}^i = -(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} \left(\sum_{j \in 2 \dots n} M_{k+1}^{ij\mu} B_{k+1}^j n_k^j \right. \\ \left. + m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^{i'} \right) + \tilde{u}_{k+1}^{ii}; i \in \{2 \dots n\}$$

$$(4.141) \quad -R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ix} + B_{k+1}^{i'} L_{k+1}^x + (B_{k+1}^{i'} (Q_{k+1}^i + L_{k+1}^{i\mu}) B_{k+1}^i \\ + R_{k+1}^{ii} - R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ii\mu} B_{k+1}^i) N_k^{ix} + \sum_{j \in 2 \dots n, j \neq i} (B_{k+1}^{i'} (Q_{k+1}^j + L_{k+1}^{j\mu}) B_{k+1}^j \\ - R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ij\mu} B_{k+1}^j) N_k^{jx} = 0; i \in \{2 \dots n\}$$

$$(4.142) \quad -R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{im\mu} A_{k+1} + B_{k+1}^{i'} L_{k+1}^{m\mu} A_{k+1} \\ + B_{k+1}^{i'} Q_{k+1}^m A_{k+1} + (B_{k+1}^{i'} (Q_{k+1}^i + L_{k+1}^{i\mu}) B_{k+1}^i + R_{k+1}^{ii} \\ - R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ii\mu} B_{k+1}^i) N_k^{im\mu} + \sum_{j \in 2 \dots n, j \neq i} (B_{k+1}^{i'} (Q_{k+1}^j + L_{k+1}^{j\mu}) B_{k+1}^j \\ - R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ij\mu} B_{k+1}^j) N_k^{jm\mu} = 0; i, m \in \{2 \dots n\}$$

$$(4.143) \quad -B_{k+1}^{i'} Q_{k+1}^1 \tilde{x}_{k+1}^i + R_{k+1}^{1i} \left(-(R_{k+1}^{ii})^{-1} B_{k+1}^{i'} (m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^{i'}) + \tilde{u}_{k+1}^{ii} \right. \\ \left. - \tilde{u}_{k+1}^{1i} \right) + B_{k+1}^{i'} l_{k+1} + (B_{k+1}^{i'} (Q_{k+1}^i + L_{k+1}^{i\mu}) B_{k+1}^i + R_{k+1}^{ii} - R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ii\mu} B_{k+1}^i) n_k^i \\ + \sum_{j \in 2 \dots n, j \neq i} (B_{k+1}^{i'} (Q_{k+1}^j + L_{k+1}^{j\mu}) B_{k+1}^j - R_{k+1}^{1i} (R_{k+1}^{ii})^{-1} B_{k+1}^{i'} M_{k+1}^{ij\mu} B_{k+1}^j) n_k^j = 0; i \in \{2 \dots n\}$$

$$(4.144) \quad M_k^{ix} = Q_k^i + A_{k+1}' [M_{k+1}^{ix} \Phi_k^x + \sum_{j \in 2 \dots n} M_{k+1}^{ij\mu} \Psi_k^{jx}]; M_T^{ix} = Q_T^i; i \in \{2 \dots n\}$$

$$(4.145) \quad M_k^{im\mu} = A_{k+1}' [M_{k+1}^{ix} \Phi_k^{m\mu} + \sum_{j \in 2 \dots n} M_{k+1}^{ij\mu} \Psi_k^{jm\mu}]; M_T^{im\mu} = 0; i, m \in \{2 \dots n\}$$

$$(4.146) \quad m_k^i = A'_{k+1} [M_{k+1}^{ix} \phi_k + \sum_{j \in 2 \dots n} M_{k+1}^{ij\mu} \psi_k^j + m_{k+1}^i - Q_{k+1}^i \tilde{x}_{k+1}^i] ; m_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.147) \quad L_k^x = Q_k^1 + A'_{k+1} [L_{k+1}^x \Phi_k^x + \sum_{j \in 2 \dots n} L_{k+1}^{j\mu} \Psi_k^{jx} + \sum_{j \in 2 \dots n} Q_{k+1}^j B_{k+1}^j N_k^{jx} \Phi_k^x] ; L_T^x = Q_T^1$$

$$(4.148) \quad L_k^{i\mu} = A'_{k+1} [L_{k+1}^x \Phi_k^{i\mu} + \sum_{j \in 2 \dots n} L_{k+1}^{j\mu} \Psi_k^{ji\mu} + Q_{k+1}^i A_{k+1} + \sum_{j \in 2 \dots n} Q_{k+1}^j B_{k+1}^j (N_k^{jx} \Phi_k^{i\mu} + N_k^{ji\mu})] ; L_T^{i\mu} = 0 ; i \in \{2 \dots n\}$$

$$(4.149) \quad l_k = A'_{k+1} [L_{k+1}^x \phi_k + \sum_{j \in 2 \dots n} L_{k+1}^{j\mu} \psi_k^j + l_{k+1} - Q_{k+1}^1 \tilde{x}_{k+1}^1 + \sum_{j \in 2 \dots n} Q_{k+1}^j B_{k+1}^j (N_k^{jx} \phi_k + n_k^j)] ; l_T = 0$$

$$(4.150) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = P_{k+1}^{ix} x_k^* + \sum_{j \in 2 \dots n} P_{k+1}^{ij\mu} \mu_k^j + \alpha_{k+1}^i$$

$$(4.151) \quad P_{k+1}^{1x} = W_{k+1}^x \Phi_k^x$$

$$(4.152) \quad P_{k+1}^{1i\mu} = W_{k+1}^x \Phi_k^{i\mu} + W_{k+1}^{i\mu} ; i \in \{2 \dots n\}$$

$$(4.153) \quad \alpha_{k+1}^1 = W_{k+1}^x \phi_k + w_{k+1}$$

$$(4.154) \quad P_{k+1}^{ix} = T_{k+1}^{ix} \Phi_k^x; i \in \{2 \dots n\}$$

$$(4.155) \quad P_{k+1}^{im\mu} = T_{k+1}^{ix} \Phi_k^{m\mu} + T_{k+1}^{im\mu}; i, m \in \{2 \dots n\}$$

$$(4.156) \quad \alpha_{k+1}^i = T_{k+1}^{ix} \phi_k + t_{k+1}^i; i \in \{2 \dots n\}$$

PROOF:²⁴

Theorem (8) can be applied to the given affine-quadratic game, since all conditions are satisfied for the given state equation (4.123) and cost functionals (4.124). Furthermore g_k^i is strictly convex in u_k^i ($i \in N$). This can be seen by applying Corollary (1) to (4.157). Therefore there has to be a unique optimal equilibrium solution.

$$(4.157) \quad \frac{\partial^2}{\partial u_k^{i^2}} g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = B_k^{i'} Q_k^i B_k^i + R_k^{ii}$$

²⁴ The first part of the proof, which is the derivation of the general optimality conditions for the game, is the same as in Theorem (9).

To obtain relations which satisfy this unique solution we have to adapt (4.32) - (4.41) to the given state equation and cost functionals. This yields

$$\begin{aligned}
 (4.158) \quad H_k^1 = & \frac{1}{2}(x_k' Q_k^1 x_k + \sum_{j \in N} u_k^{j'} R_k^{1j} u_k^j) + \frac{1}{2}(\tilde{x}_k^{1'} Q_k^1 \tilde{x}_k^1 + \sum_{j \in N} \tilde{u}_k^{1j'} R_k^{1j} \tilde{u}_k^{1j}) \\
 & - \tilde{x}_k^{1'} Q_k^1 x_k - \sum_{j \in N} \tilde{u}_k^{1j'} R_k^{1j} u_k^j + \lambda_k' (A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k) \\
 & + \sum_{j \in 2 \dots n} \mu_{k-1}^{j'} A_k' [p_k^j + Q_k^j (x_k^* - \tilde{x}_k^j)] \\
 & + \sum_{j \in 2 \dots n} v_{k-1}^{j'} (B_k^{j'} Q_k^j x_k + R_k^{jj} u_k^j - B_k^{j'} Q_k^j \tilde{x}_k^{j'} - R_k^{jj} \tilde{u}_k^{jj} + B_k^{j'} p_k^j)
 \end{aligned}$$

$$\begin{aligned}
 (4.159) \quad H_k^i = & \frac{1}{2}(x_k' Q_k^i x_k + \sum_{j \in N} u_k^{j'} R_k^{ij} u_k^j) + \frac{1}{2}(\tilde{x}_k^{i'} Q_k^i \tilde{x}_k^i + \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} \tilde{u}_k^{ij}) \\
 & - \tilde{x}_k^{i'} Q_k^i x_k - \sum_{j \in N} \tilde{u}_k^{ij'} R_k^{ij} u_k^j + p_k^{i'} (A_k x_{k-1} + \sum_{j \in N} B_k^j u_k^j + s_k); i \in \{2 \dots n\}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial}{\partial u_k^1} (4.158) = 0 \Rightarrow & B_k^{1'} Q_k^1 (x_k^* - \tilde{x}_k^1) + R_k^{11} (u_k^{1*} - \tilde{u}_k^{11}) + B_k^{1'} \lambda_k \\
 & + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j A_k \mu_{k-1}^j + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j B_k^j v_{k-1}^j = 0
 \end{aligned}$$

$$\begin{aligned}
 (4.160) \quad u_k^{1*} = & -(R_k^{11})^{-1} (B_k^{1'} Q_k^1 (x_k^* - \tilde{x}_k^1) + B_k^{1'} \lambda_k \\
 & + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j A_k \mu_{k-1}^j + \sum_{j \in 2 \dots n} B_k^{1'} Q_k^j B_k^j v_{k-1}^j) + \tilde{u}_k^{11}
 \end{aligned}$$

$$\begin{aligned}
 (4.161) \quad \frac{\partial}{\partial u_k^i} (4.158) = 0 \Rightarrow & B_k^{i'} Q_k^1 (x_k^* - \tilde{x}_k^1) + R_k^{1i} (u_k^{1*} - \tilde{u}_k^{11}) + B_k^{i'} \lambda_k \\
 & + \sum_{j \in 2 \dots n} B_k^{i'} Q_k^j A_k \mu_{k-1}^j + (B_k^{i'} Q_k^j B_k^j + R_k^{ii}) v_{k-1}^j + \sum_{j \in 2 \dots n, j \neq i} B_k^{i'} Q_k^j B_k^j v_{k-1}^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

$$(4.162) \quad \lambda_{k-1} = A'_k Q_k^1 (x_k^* - \tilde{x}_k^1) + A'_k \lambda_k \\ + \sum_{j \in 2 \dots n} A'_k Q_k^j A_k \mu_{k-1}^j + \sum_{j \in 2 \dots n} A'_k Q_k^j B_k^j v_{k-1}^j; \lambda_T = 0$$

$$(4.163) \quad \mu_k^i = A_k \mu_{k-1}^i + B_k^i v_{k-1}^i; \mu_0^i = 0; i \in \{2 \dots n\}$$

$$\frac{\partial}{\partial u_i}(4.159) = 0 \Rightarrow B_k^{i'} Q_k^i x_k^* + R_k^{ii} u_k^{i*} - B_k^{i'} Q_k^i \tilde{x}_k^{i'} - R_k^{ii} \tilde{u}_k^{ii} + B_k^{i'} p_k^i = 0; i \in \{2 \dots n\}$$

$$(4.164) \quad u_k^{i*} = -(R_k^{ii})^{-1} B_k^{i'} (Q_k^i (x_k^* - \tilde{x}_k^{i'}) + p_k^i) + \tilde{u}_k^{ii}$$

$$(4.165) \quad p_{k-1}^{i*} = A'_k [p_k^i + Q_k^i (x_k^* - \tilde{x}_k^i)]; p_T^i = 0; i \in \{2 \dots n\}$$

$$(4.166) \quad x_k^* = A_k x_{k-1}^* + \sum_{j \in N} B_k^j u_k^{j*} + s_k; x_0^* = x_0$$

In the following induction argument, we will give proof that (4.167) and (4.168) are valid and the recursive relations for M_k^{ix} , $M_k^{ij\mu}$, m_k^i , L_k^x , $L_k^{i\mu}$ and l_k ($i, j \in \{2 \dots n\}$) (stated in the above theorem) are correct.

$$(4.167) \quad p_k^i = (M_k^{ix} - Q_k^i) x_k^* + \sum_{j \in 2 \dots n} M_k^{ij\mu} \mu_k^j + m_k^i; i \in \{2 \dots n\}$$

$$(4.168) \quad \lambda_k = (L_k^x - Q_k^1) x_k^* + \sum_{j \in 2 \dots n} L_k^{j\mu} \mu_k^j + l_k$$

Basis:

The induction starts at $k = T$. First we make use of the general optimality conditions for p_k^i and λ_k at stage T .

$$(4.165)_{k=T} \quad p_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.162)_{k=T} \quad \lambda_T = 0$$

Now we can substitute p_T^i and λ_T with functions affinely dependent on $(x_T^*, \mu_T^2, \dots, \mu_T^n)$.

$$(4.167)_{k=T} \quad p_T^i = (M_T^{ix} - Q_T^i)x_T^* + \sum_{j \in 2 \dots n} M_T^{ij\mu} \mu_T^j + m_T^i ; i \in \{2 \dots n\}$$

$$(M_T^{ix} - Q_T^i)x_T^* + \sum_{j \in 2 \dots n} M_T^{ij\mu} \mu_T^j + m_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.168)_{k=T} \quad \lambda_T = (L_T^x - Q_T^1)x_T^* + \sum_{j \in 2 \dots n} L_T^{j\mu} \mu_T^j + l_T$$

$$(L_T^x - Q_T^1)x_T^* + \sum_{j \in 2 \dots n} L_T^{j\mu} \mu_T^j + l_T = 0$$

Comparing coefficients gives

$$(4.144)_{k=T} \quad M_T^{ix} = Q_T^i ; i \in \{2 \dots n\}$$

$$(4.145)_{k=T} \quad M_T^{ij\mu} = 0 ; i, j \in \{2 \dots n\}$$

$$(4.146)_{k=T} \quad m_T^i = 0 ; i \in \{2 \dots n\}$$

$$(4.147)_{k=T} \quad L_T^x = Q_T^1$$

$$(4.148)_{k=T} \quad L_T^{j\mu} = 0 ; j \in \{2 \dots n\}$$

$$(4.149)_{k=T} \quad l_T = 0$$

Inductive step:

As induction hypotheses, the system of equations (4.167) and equation (4.168) are assumed to be true at stage $l+2$. Now we have to prove that these equations are fulfilled at stage $l+1$ and determine the corresponding recursive relations for M_l^{ix} , $M_l^{ij\mu}$, m_l^i , L_l^x , $L_l^{i\mu}$ and l_l ($i, j \in \{2, \dots, n\}$).

$$(4.167)_{k=l+2} \quad p_{l+1}^i = (M_{l+1}^{ix} - Q_{l+1}^i)x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \mu_{l+1}^j + m_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.168)_{k=l+2} \quad \lambda_{l+1} = (L_{l+1}^x - Q_{l+1}^1)x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \mu_{l+1}^j + l_{l+1}$$

First the induction hypotheses are used in the general optimality conditions for p_k^i and λ_k at stage $l+1$.

$$(4.165)_{k=l+1} \quad p_l^{i*} = A'_{l+1}[p_{l+1}^i + Q_{l+1}^i(x_{l+1}^* - \tilde{x}_{l+1}^i)] ; i \in \{2 \dots n\}$$

$$p_l^{i*} = A'_{l+1}[M_{l+1}^{ix}x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu}\mu_{l+1}^j + m_{l+1}^i - Q_{l+1}^i\tilde{x}_{l+1}^i] ; i \in \{2 \dots n\}$$

$$(4.162)_{k=l+1} \quad \lambda_l = A'_{l+1}Q_{l+1}^1(x_{l+1}^* - \tilde{x}_{l+1}^1) + A'_{l+1}\lambda_{l+1} \\ + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}\mu_l^j + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j v_l^j$$

$$\lambda_l = A'_{l+1}(L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu}\mu_{l+1}^j + l_{l+1}) - A'_{l+1}Q_{l+1}^1\tilde{x}_{l+1}^1 \\ + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1}\mu_l^j + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j v_l^j$$

To complete the inductive step, we have to show that the p_l^i and λ_l can be written as affine functions of the variables $(x_l^*, \mu_l^2, \dots, \mu_l^n)$. Therefore, interrelations between x_{l+1}^* and $(x_l^*, \mu_l^2, \dots, \mu_l^n)$ and between μ_{l+1}^i and $(x_l^*, \mu_l^2, \dots, \mu_l^n)$ ($i \in \{2, \dots, n\}$) that do not depend on the controls of the players nor on costate $(p_{l+1}^i, \lambda_{l+1}^i)$ or co-control (v_l^j) variables have to be deduced. To do so, first we have to substitute $u_{l+1}^{1*}, \dots, u_{l+1}^{n*}$ in the equation stated below for the evolution of the optimal state vector x_{l+1}^* by terms that are affine in $(x_l^*, x_{l+1}^*, \mu_l^2, \dots, \mu_l^n)$ and furthermore only contain $M_{l+1}^{ix}, M_{l+1}^{ij\mu}, m_{l+1}^i, L_{l+1}^x, L_{l+1}^{i\mu}, l_{l+1}$ and matrices and vectors given by the game definition.

$$(4.166)_{k=l+1} \quad x_{l+1}^* = A_{l+1}x_l^* + \sum_{j \in N} B_{l+1}^j u_{l+1}^{j*} + s_{l+1}$$

In the first instance the optimality condition for u_{l+1}^* ($i \in \{2 \dots n\}$) can be rewritten with the help of (4.167) $_{k=l+2}$, which are the induction hypotheses for p_{l+1}^i .

$$(4.164)_{k=l+1} \quad u_{l+1}^{i*} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (Q_{l+1}^i (x_{l+1}^* - \tilde{x}_{l+1}^{i'}) + p_{l+1}^i) + \tilde{u}_{l+1}^{ii}; i \in \{2 \dots n\}$$

$$\begin{aligned} u_{l+1}^{i*} = & -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \mu_{l+1}^j \\ & + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}; i \in \{2 \dots n\} \end{aligned}$$

As a next step the stage indices of μ^i have to be reduced from $l+1$ to l with the help of the optimality conditions for μ_{l+1}^i .

$$(4.163)_{k=l+1} \quad \mu_{l+1}^i = A_{l+1} \mu_l^i + B_{l+1}^i v_l^i; i \in \{2 \dots n\}$$

$$\begin{aligned} (4.169) \quad u_{l+1}^{i*} = & -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \\ & (A_{l+1} \mu_l^j + B_{l+1}^j v_l^j) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}; i \in \{2 \dots n\} \end{aligned}$$

Moreover, we have to substitute v_l^j ($j \in \{2 \dots n\}$). For that purpose, v_l^j has to be explicated from (4.161) $_{l+1}$. As a start u_{l+1}^{i*} is replaced using (4.169) and the v_l^j ($j \in \{2, \dots, n\}$) are abstracted in one term.

$$\begin{aligned} (4.161)_{k=l+1} \quad & B_{l+1}^{i'} Q_{l+1}^1 (x_{l+1}^* - \tilde{x}_{l+1}^i) + R_{l+1}^{1i} (u_{l+1}^{i*} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} \lambda_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j \\ & + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) v_l^i + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j v_l^j = 0; i \in \{2 \dots n\} \end{aligned}$$

$$\begin{aligned}
 & B_{l+1}^{i'} Q_{l+1}^1 (x_{l+1}^* - \tilde{x}_{l+1}^i) + R_{l+1}^{1i} (- (R_{l+1}^{ii})^{-1} B_{l+1}^{i'}) \\
 & \quad (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} (A_{l+1} \mu_l^j + B_{l+1}^j v_l^j) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) \\
 & \quad + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} \lambda_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j \\
 & \quad + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii}) v_l^i + \sum_{j \in 2 \dots n, j \neq i} B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j v_l^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

As a next step λ_{l+1} is substituted with the help of (4.168)_{k=l+1}.

$$\begin{aligned}
 & -B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^i + R_{l+1}^{1i} (- (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} A_{l+1} \mu_l^j + m_{l+1}^i \\
 & \quad - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} (L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \mu_{l+1}^j + l_{l+1}) \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) v_l^i \\
 & + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) v_l^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

Eventually the μ_{l+1}^i are replaced making use of (4.163)_{k=l+1} and the v_l^j ($j \in \{2, \dots, n\}$) are again abstracted in one term.

$$\begin{aligned}
 & -B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^i + R_{l+1}^{1i} (- (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} A_{l+1} \mu_l^j + m_{l+1}^i \\
 & \quad - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} (L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} (A_{l+1} \mu_l^j + B_{l+1}^j v_l^j) + l_{l+1}) \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j + (B_{l+1}^{i'} Q_{l+1}^i B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) v_l^i \\
 & + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} Q_{l+1}^j B_{l+1}^j - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) v_l^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

$$\begin{aligned}
 & -B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^i + R_{l+1}^{1i} (-(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} A_{l+1} \mu_l^j + m_{l+1}^i \\
 & - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} (L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} A_{l+1} \mu_l^j + l_{l+1}) \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j + (B_{l+1}^{i'} (Q_{l+1}^i + L_{l+1}^{i\mu}) B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) v_l^i \\
 & + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} (Q_{l+1}^j + L_{l+1}^{j\mu}) B_{l+1}^j - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) v_l^j = 0; i \in \{2 \dots n\}
 \end{aligned}$$

The above equations only contain constant expressions and terms linear in x_{l+1} or μ_l^j ($j \in \{2, \dots, n\}$). This fact justifies the following substitutions:

$$(4.170) \quad v_l^i = N_l^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} N_l^{ij\mu} \mu_l^j + n_l^i; i \in \{2 \dots n\}$$

$$\begin{aligned}
 (4.171) \quad & -B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^i + R_{l+1}^{1i} (-(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} A_{l+1} \mu_l^j \\
 & + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} (L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} A_{l+1} \mu_l^j + l_{l+1}) \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{i'} Q_{l+1}^j A_{l+1} \mu_l^j + (B_{l+1}^{i'} (Q_{l+1}^i + L_{l+1}^{i\mu}) B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) \\
 & (N_l^{ix} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{im\mu} \mu_l^m + n_l^i) + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} (Q_{l+1}^j + L_{l+1}^{j\mu}) B_{l+1}^j \\
 & - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j) = 0; i \in \{2 \dots n\}
 \end{aligned}$$

Comparing coefficients gives the following systems of equations that are assumed to admit unique solutions N_l^{ix} , $N_l^{im\mu}$ and n_l^i ($i, m \in \{2, \dots, n\}$).

$$(4.171)_{x_{l+1}^*} = (4.141)_{k=l} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ix} + B_{l+1}^{i'} L_{l+1}^x + (B_{l+1}^{i'} (Q_{l+1}^i + L_{l+1}^{i\mu}) B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) N_l^{ix} + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} (Q_{l+1}^j + L_{l+1}^{j\mu}) B_{l+1}^j - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) N_l^{jx} = 0; i \in \{2 \dots n\}$$

$$(4.171)_{\mu_l^m} = (4.142)_{k=l} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{im\mu} A_{l+1} + B_{l+1}^{i'} L_{l+1}^{m\mu} A_{l+1} + B_{l+1}^{i'} Q_{l+1}^m A_{l+1} + (B_{l+1}^{i'} (Q_{l+1}^i + L_{l+1}^{i\mu}) B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) N_l^{im\mu} + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} (Q_{l+1}^j + L_{l+1}^{j\mu}) B_{l+1}^j - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) N_l^{jm\mu} = 0; i, m \in \{2 \dots n\}$$

$$(4.171)_{const.} = (4.143)_{k=l} - B_{l+1}^{i'} Q_{l+1}^1 \tilde{x}_{l+1}^i + R_{l+1}^{1i} (- (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} - \tilde{u}_{l+1}^{1i}) + B_{l+1}^{i'} l_{l+1} + (B_{l+1}^{i'} (Q_{l+1}^i + L_{l+1}^{i\mu}) B_{l+1}^i + R_{l+1}^{ii} - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ii\mu} B_{l+1}^i) n_l^i + \sum_{j \in 2 \dots n, j \neq i} (B_{l+1}^{i'} (Q_{l+1}^j + L_{l+1}^{j\mu}) B_{l+1}^j - R_{l+1}^{1i} (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} M_{l+1}^{ij\mu} B_{l+1}^j) n_l^j = 0; i \in \{2 \dots n\}$$

Using the elaborated relation for v_l^j ($j \in \{2 \dots n\}$) in (4.169) yields

$$u_{l+1}^{i*} = - (R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} (A_{l+1} \mu_l^j + B_{l+1}^j (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii}; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$(4.172) \quad u_{l+1}^{i*} = T_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} T_{l+1}^{ij\mu} \mu_l^j + t_{l+1}^i; i \in \{2 \dots n\}$$

$$\begin{aligned}
 (4.173) \quad T_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} T_{l+1}^{ij\mu} \mu_l^j + t_{l+1}^i = & -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} x_{l+1}^* \\
 & + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} (A_{l+1} \mu_l^j + B_{l+1}^j (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)) \\
 & + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} ; i \in \{2 \dots n\}
 \end{aligned}$$

By comparing coefficients it follows that

$$(4.173)_{x_{l+1}^*} = (4.138)_{k=l} \quad T_{l+1}^{ix} = -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{ix} + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} B_{l+1}^j N_l^{jx}) ; i \in \{2 \dots n\}$$

$$\begin{aligned}
 (4.173)_{\mu_l^m} = (4.139)_{k=l} \quad T_{l+1}^{im\mu} = & -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (M_{l+1}^{im\mu} A_{l+1} \\
 & + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} B_{l+1}^j N_l^{jm\mu}) ; i, m \in \{2 \dots n\}
 \end{aligned}$$

$$\begin{aligned}
 (4.173)_{const.} = (4.140)_{k=l} \quad t_{l+1}^i = & -(R_{l+1}^{ii})^{-1} B_{l+1}^{i'} (\sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} B_{l+1}^j n_l^j \\
 & + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^{i'}) + \tilde{u}_{l+1}^{ii} ; i \in \{2 \dots n\}
 \end{aligned}$$

The optimality condition for u_{l+1}^{1*} can be rewritten with the help of $(4.168)_{k=l+1}$ and $(4.163)_{k=l+1}$. These are the induction hypothesis for λ_{l+1} and the conditions for the optimal evolution of μ_{l+1}^i .

$$\begin{aligned}
 (4.160)_{k=l+1} \quad u_{l+1}^{1*} = & -(R_{l+1}^{11})^{-1} (B_{l+1}^{1'} Q_{l+1}^1 (x_{l+1}^* - \tilde{x}_{l+1}^1) + B_{l+1}^{1'} \lambda_{l+1} \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j v_l^j) + \tilde{u}_{l+1}^{11}
 \end{aligned}$$

$$u_{l+1}^{1*} = -(R_{l+1}^{11})^{-1}(-B_{l+1}^{1'}Q_{l+1}^1\tilde{x}_{l+1}^1 + B_{l+1}^{1'}(L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \mu_{l+1}^j + l_{l+1})) \\ + \sum_{j \in 2 \dots n} B_{l+1}^{1'}Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} B_{l+1}^{1'}Q_{l+1}^j B_{l+1}^j v_l^j) + \tilde{u}_{l+1}^{11}$$

$$u_{l+1}^{1*} = -(R_{l+1}^{11})^{-1}(-B_{l+1}^{1'}Q_{l+1}^1\tilde{x}_{l+1}^1 + B_{l+1}^{1'}(L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} (A_{l+1} \mu_l^j + B_{l+1}^j v_l^j) \\ + l_{l+1})) + \sum_{j \in 2 \dots n} B_{l+1}^{1'}Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} B_{l+1}^{1'}Q_{l+1}^j B_{l+1}^j v_l^j) + \tilde{u}_{l+1}^{11}$$

Now v_l^i ($i \in \{2, \dots, n\}$) can be replaced by the relations deduced above stated in (4.170).

$$u_{l+1}^{1*} = -(R_{l+1}^{11})^{-1}(-B_{l+1}^{1'}Q_{l+1}^1\tilde{x}_{l+1}^1 + B_{l+1}^{1'}(L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} (A_{l+1} \mu_l^j \\ + B_{l+1}^j (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)) + l_{l+1})) + \sum_{j \in 2 \dots n} B_{l+1}^{1'}Q_{l+1}^j A_{l+1} \mu_l^j \\ + \sum_{j \in 2 \dots n} B_{l+1}^{1'}Q_{l+1}^j B_{l+1}^j (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)) + \tilde{u}_{l+1}^{11}$$

The structure of the above equation justifies the following substitution

$$(4.174) \quad u_{l+1}^{1*} = W_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1}$$

$$\begin{aligned}
 (4.175) \quad W_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1} = & -(R_{l+1}^{11})^{-1} (-B_{l+1}^{1'} Q_{l+1}^1 \tilde{x}_{l+1}^1 \\
 & + B_{l+1}^{1'} (L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} (A_{l+1} \mu_l^j + B_{l+1}^j (N_l^{jx} x_{l+1}^* \\
 & + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)) + l_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j A_{l+1} \mu_l^j \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)) + \tilde{u}_{l+1}^{11}
 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}
 (4.175)_{x_{l+1}^*} = (4.135)_{k=l} \quad W_{l+1}^x = & -(R_{l+1}^{11})^{-1} (B_{l+1}^{1'} (L_{l+1}^x + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} B_{l+1}^j N_l^{jx}) \\
 & + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j N_l^{jx})
 \end{aligned}$$

$$\begin{aligned}
 (4.175)_{\mu_l^m} = (4.136)_{k=l} \quad W_{l+1}^{m\mu} = & -(R_{l+1}^{11})^{-1} (B_{l+1}^{1'} (L_{l+1}^{m\mu} A_{l+1} + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} B_{l+1}^j N_l^{jm\mu}) \\
 & + B_{l+1}^{1'} Q_{l+1}^m A_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j N_l^{jm\mu}) ; m \in \{2 \dots n\}
 \end{aligned}$$

$$\begin{aligned}
 (4.175)_{const.} = (4.137)_{k=l} \quad w_{l+1} = & -(R_{l+1}^{11})^{-1} (-B_{l+1}^{1'} Q_{l+1}^1 \tilde{x}_{l+1}^1 + B_{l+1}^{1'} (\sum_{j \in 2 \dots n} L_{l+1}^{j\mu} B_{l+1}^j n_l^j \\
 & + l_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^{1'} Q_{l+1}^j B_{l+1}^j n_l^j) + \tilde{u}_{l+1}^{11}
 \end{aligned}$$

At this point it is possible to replace the control variables in the optimal state equation at stage $l+1$ by terms affine in $(x_{l+1}^*, \mu_l^2, \dots, \mu_l^n)$.

$$(4.166)_{k=l+1} \quad x_{l+1}^* = A_{l+1} x_l^* + B_{l+1}^1 u_{l+1}^{1*} + \sum_{j \in \{2 \dots n\}} B_{l+1}^j u_{l+1}^{j*} + s_{l+1}$$

$$\begin{aligned}
 x_{l+1}^* &= A_{l+1}x_l^* + B_{l+1}^1(W_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1}) \\
 &\quad + \sum_{j \in 2 \dots n} B_{l+1}^j(T_{l+1}^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} T_{l+1}^{jm\mu} \mu_l^m + t_{l+1}^j) + s_{l+1}
 \end{aligned}$$

Making x_{l+1}^* explicit yields

$$\begin{aligned}
 (I - B_{l+1}^1 W_{l+1}^x - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^{jx}) x_{l+1}^* &= A_{l+1}x_l^* + B_{l+1}^1(\sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j \\
 &\quad + w_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^j(\sum_{m \in 2 \dots n} T_{l+1}^{jm\mu} \mu_l^m + t_{l+1}^j) + s_{l+1}
 \end{aligned}$$

$$\begin{aligned}
 x_{l+1}^* &= (I - B_{l+1}^1 W_{l+1}^x - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^{jx})^{-1} (A_{l+1}x_l^* + B_{l+1}^1(\sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j \\
 &\quad + w_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^j(\sum_{m \in 2 \dots n} T_{l+1}^{jm\mu} \mu_l^m + t_{l+1}^j) + s_{l+1})
 \end{aligned}$$

The structure of the above equation justifies the following substitution

$$(4.126)_{k=l} \quad x_{l+1}^* = \Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l$$

$$\begin{aligned}
 (4.176) \quad (I - B_{l+1}^1 W_{l+1}^x - \sum_{j \in \{2 \dots n\}} B_{l+1}^j T_{l+1}^{jx})(\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) &= A_{l+1}x_l^* \\
 &\quad + B_{l+1}^1(\sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1}) + \sum_{j \in 2 \dots n} B_{l+1}^j(\sum_{m \in 2 \dots n} T_{l+1}^{jm\mu} \mu_l^m + t_{l+1}^j) + s_{l+1}
 \end{aligned}$$

By comparing coefficients it follows that

$$(4.176)_{x_l^*} = (4.127)_{k=l} \quad \Phi_l^x = (I - B_{l+1}^1 W_{l+1}^x - \sum_{j \in 2 \dots n} B_{l+1}^j T_{l+1}^{jx})^{-1} A_{l+1}$$

$$(4.176)_{\mu_l^i} = (4.128)_{k=l} \quad \Phi_l^{i\mu} = (I - B_{l+1}^1 W_{l+1}^x - \sum_{j \in 2 \dots n} B_{l+1}^j T_{l+1}^{jx})^{-1} \\ (B_{l+1}^1 W_{l+1}^{i\mu} + \sum_{j \in 2 \dots n} B_{l+1}^j T_{l+1}^{ji\mu}) ; i \in \{2 \dots n\}$$

$$(4.176)_{const.} = (4.129)_{k=l} \quad \phi_l = (I - B_{l+1}^1 W_{l+1}^x - \sum_{j \in 2 \dots n} B_{l+1}^j T_{l+1}^{jx})^{-1} \\ (B_{l+1}^1 w_{l+1} + \sum_{j \in 2 \dots n} B_{l+1}^j t_{l+1}^j + s_{l+1})$$

As a next step affine relations between $(x_l^*, \mu_{l+1}^i, \mu_l^2, \dots, \mu_l^n)$ ($i \in \{2, \dots, n\}$) are derived. Therefore v_l^i is substituted in $(4.163)_{k=l+1}$ by making use of (4.170).

$$(4.163)_{k=l+1} \quad \mu_{l+1}^i = A_{l+1} \mu_l^i + B_{l+1}^i v_l^i ; i \in \{2 \dots n\}$$

$$\mu_{l+1}^i = A_{l+1} \mu_l^i + B_{l+1}^i (N_l^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} N_l^{ij\mu} \mu_l^j + n_l^i) ; i \in \{2 \dots n\}$$

Using $(4.126)_{k=l}$ yields

$$\mu_{l+1}^i = A_{l+1} \mu_l^i + B_{l+1}^i (N_l^{ix} (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} N_l^{ij\mu} \mu_l^j + n_l^i) ; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$(4.130)_{k=l} \quad \mu_{l+1}^i = \Psi_l^{ix} x_l^* + \sum_{j \in 2 \dots n} \Psi_l^{ij\mu} \mu_l^j + \psi_l^i ; i \in \{2 \dots n\}$$

$$(4.177) \quad \Psi_l^x x_l^* + \sum_{j \in 2 \dots n} \Psi_l^{j\mu} \mu_l^j + \psi_l = A_{l+1} \mu_l^i + B_{l+1}^i (N_l^{ix} (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} N_l^{ij\mu} \mu_l^j + n_l^i); i \in \{2 \dots n\}$$

Comparing coefficients gives

$$(4.177)_{x_l^*} = (4.131)_{k=l} \quad \Psi_l^{ix} = B_{l+1}^i N_l^{ix} \Phi_l^x; i \in \{2 \dots n\}$$

$$(4.177)_{\mu_l^i} = (4.132)_{k=l} \quad \Psi_l^{ii} = A_{l+1} + B_{l+1}^i (N_l^{ix} \Phi_l^{i\mu} + N_l^{ii\mu}); i \in \{2 \dots n\}$$

$$(4.177)_{\mu_l^m} = (4.133)_{k=l} \quad \Psi_l^{im} = B_{l+1}^i (N_l^{ix} \Phi_l^{m\mu} + N_l^{im\mu}); i, m \in \{2 \dots n\}; m \neq i$$

$$(4.177)_{const} = (4.134)_{k=l} \quad \psi_l^i = B_{l+1}^i (N_l^{ix} \phi_l + n_l^i); i \in \{2 \dots n\}$$

Now it is possible to finish the inductive step of p_l^i and λ_l . As a start $(4.126)_{k=l}$ and $(4.130)_{k=l}$ are used to continue the derivation of p_l^i .

$$p_l^{i*} = A_{l+1}' [M_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \mu_{l+1}^j + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]; i \in \{2 \dots n\}$$

$$p_l^{i*} = A_{l+1}' [M_{l+1}^{ix} (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} (\Psi_l^{ix} x_l^* + \sum_{m \in 2 \dots n} \Psi_l^{jm\mu} \mu_l^m + \psi_l^j) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$(4.167)_{k=l+1} \quad p_l^i = (M_l^{ix} - Q_l^i)x_l^* + \sum_{j \in 2 \dots n} M_l^{ij\mu} \mu_l^j + m_l^i; i \in \{2 \dots n\}$$

$$(4.178) \quad (M_l^{ix} - Q_l^i)x_l^* + \sum_{j \in 2 \dots n} M_l^{ij\mu} \mu_l^j + m_l^i = A'_{l+1} [M_{l+1}^{ix} (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} (\Psi_l^{jx} x_l^* + \sum_{m \in 2 \dots n} \Psi_l^{jm\mu} \mu_l^m + \psi_l^j) + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]; i \in \{2 \dots n\}$$

By comparing coefficients it follows that

$$(4.178)_{x_l^*} = (4.144)_{k=l} \quad M_l^{ix} = Q_l^i + A'_{l+1} [M_{l+1}^{ix} \Phi_l^x + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \Psi_l^{jx}]; i \in \{2 \dots n\}$$

$$(4.178)_{\mu_l^m} = (4.145)_{k=l} \quad M_l^{im\mu} = A'_{l+1} [M_{l+1}^{ix} \Phi_l^{m\mu} + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \Psi_l^{jm\mu}]; i, m \in \{2 \dots n\}$$

$$(4.178)_{const.} = (4.146)_{k=l} \quad m_l^i = A'_{l+1} [M_{l+1}^{ix} \phi_l + \sum_{j \in 2 \dots n} M_{l+1}^{ij\mu} \psi_l^j + m_{l+1}^i - Q_{l+1}^i \tilde{x}_{l+1}^i]; i \in \{2 \dots n\}$$

As a next step $(4.126)_{k=l}$ is used twice and $(4.130)_{k=l}$ and (4.170) are applied once to continue the derivation of λ_l .

$$\begin{aligned} \lambda_l = & A'_{l+1} (L_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \mu_{l+1}^j + l_{l+1}) - A'_{l+1} Q_{l+1}^1 \tilde{x}_{l+1}^1 \\ & + \sum_{j \in 2 \dots n} A'_{l+1} Q_{l+1}^j A_{l+1} \mu_l^j + \sum_{j \in 2 \dots n} A'_{l+1} Q_{l+1}^j B_{l+1}^j v_l^j \end{aligned}$$

$$\begin{aligned}
 \lambda_l = & A'_{l+1}(L_{l+1}^x(\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu}(\Psi_l^{jx} x_l^* + \sum_{m \in 2 \dots n} \Psi_l^{jm\mu} \mu_l^m + \psi_l^j) \\
 & + l_{l+1}) - A'_{l+1}Q_{l+1}^1 \tilde{x}_{l+1}^1 + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1} \mu_l^j \\
 & + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{jx} x_{l+1}^* + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)
 \end{aligned}$$

$$\begin{aligned}
 \lambda_l = & A'_{l+1}(L_{l+1}^x(\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu}(\Psi_l^{jx} x_l^* + \sum_{m \in 2 \dots n} \Psi_l^{jm\mu} \mu_l^m + \psi_l^j) \\
 & + l_{l+1}) - A'_{l+1}Q_{l+1}^1 \tilde{x}_{l+1}^1 + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1} \mu_l^j \\
 & + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{jx}(\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)
 \end{aligned}$$

The structure of the above equation justifies the following substitution

$$(4.168)_{k=l+1} \quad \lambda_l = (L_l^x - Q_l^1)x_l^* + \sum_{j \in 2 \dots n} L_l^{j\mu} \mu_l^j + l_l$$

$$\begin{aligned}
 (4.179) \quad & (L_l^x - Q_l^1)x_l^* + \sum_{j \in 2 \dots n} L_l^{j\mu} \mu_l^j + l_l = A'_{l+1}(L_{l+1}^x(\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) \\
 & + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu}(\Psi_l^{jx} x_l^* + \sum_{m \in 2 \dots n} \Psi_l^{jm\mu} \mu_l^m + \psi_l^j) + l_{l+1}) - A'_{l+1}Q_{l+1}^1 \tilde{x}_{l+1}^1 + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j A_{l+1} \mu_l^j \\
 & + \sum_{j \in 2 \dots n} A'_{l+1}Q_{l+1}^j B_{l+1}^j (N_l^{jx}(\Phi_l^x x_l^* + \sum_{m \in 2 \dots n} \Phi_l^{m\mu} \mu_l^m + \phi_l) + \sum_{m \in 2 \dots n} N_l^{jm\mu} \mu_l^m + n_l^j)
 \end{aligned}$$

Comparing coefficients gives

$$(4.179)_{x_l^*} = (4.147)_{k=l} \quad L_l^x = Q_l^1 + A'_{l+1}[L_{l+1}^x \Phi_l^x + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \Psi_l^{jx} + \sum_{j \in 2 \dots n} Q_{l+1}^j B_{l+1}^j N_l^{jx} \Phi_l^x]$$

$$(4.179)_{\mu_l^i} = (4.148)_{k=l} \quad L_l^{i\mu} = A'_{l+1} [L_{l+1}^x \Phi_l^{i\mu} + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \Psi_l^{ji\mu} + Q_{l+1}^i A_{l+1} \\ + \sum_{j \in 2 \dots n} Q_{l+1}^j B_{l+1}^j (N_l^{jx} \Phi_l^{i\mu} + N_l^{ji\mu})] ; i \in \{2 \dots n\}$$

$$(4.179)_{const.} = (4.149)_{k=l} \quad l_l = A'_{l+1} [L_{l+1}^x \phi_l + \sum_{j \in 2 \dots n} L_{l+1}^{j\mu} \psi_l^j + l_{l+1} \\ - Q_{l+1}^1 \tilde{x}_{l+1}^1 + \sum_{j \in 2 \dots n} Q_{l+1}^j B_{l+1}^j (N_l^{jx} \phi_l + n_l^j)]$$

At this point the inductive step and hence the induction argument is completed. But we will try to transform $u_{l+1}^{1*}, \dots, u_{l+1}^{n*}$ so that their evolution depends affinely on $(x_l^*, \mu_l^2, \dots, \mu_l^n)$ and so therefore their algorithmic computation is straightforward.

Let us start with u_{l+1}^{1*} by applying $(4.126)_{k=l}$ to (4.174)

$$u_{l+1}^{1*} = W_{l+1}^x x_{l+1}^* + \sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1}$$

$$u_{l+1}^{1*} = W_{l+1}^x (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1}$$

The structure of the above equation justifies the following substitution

$$u_{l+1}^{1*} = P_{l+1}^{1x} x_l + \sum_{j \in 2 \dots n} P_{l+1}^{1j\mu} \mu_l^j + \alpha_{l+1}^1$$

$$(4.180) \quad P_{l+1}^{1x} x_l + \sum_{j \in 2 \dots n} P_{l+1}^{1j\mu} \mu_l^j + \alpha_{l+1}^1 = W_{l+1}^x (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} W_{l+1}^{j\mu} \mu_l^j + w_{l+1}$$

By comparing coefficients it follows that

$$(4.180)_{x_l^*} = (4.151)_{k=l} \quad P_{l+1}^{1x} = W_{l+1}^x \Phi_l^x$$

$$(4.180)_{\mu_l^i} = (4.152)_{k=l} \quad P_{l+1}^{1i\mu} = W_{l+1}^x \Phi_l^{i\mu} + W_{l+1}^{i\mu} ; i \in \{2 \dots n\}$$

$$(4.180)_{const.} = (4.153)_{k=l} \quad \alpha_{l+1}^1 = W_{l+1}^x \phi_l + w_{l+1}$$

Finally $(4.126)_{k=l}$ is used in (4.172).

$$u_{l+1}^{i*} = T_{l+1}^{ix} x_{l+1}^* + \sum_{j \in 2 \dots n} T_{l+1}^{ij\mu} \mu_l^j + t_{l+1}^i ; i \in \{2 \dots n\}$$

$$u_{l+1}^{i*} = T_{l+1}^{ix} (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} T_{l+1}^{ij\mu} \mu_l^j + t_{l+1}^i ; i \in \{2 \dots n\}$$

The structure of the above equations justifies the following substitutions

$$u_{l+1}^{i*} = P_{l+1}^{ix} x_l + \sum_{j \in 2 \dots n} P_{l+1}^{ij\mu} \mu_l^j + \alpha_{l+1}^i ; i \in \{2 \dots n\}$$

$$(4.181) \quad P_{l+1}^{ix} x_l + \sum_{j \in 2 \dots n} P_{l+1}^{ij\mu} \mu_l^j + \alpha_{l+1}^i = T_{l+1}^{ix} (\Phi_l^x x_l^* + \sum_{j \in 2 \dots n} \Phi_l^{j\mu} \mu_l^j + \phi_l) + \sum_{j \in 2 \dots n} T_{l+1}^{ij\mu} \mu_l^j + t_{l+1}^i ; i \in \{2 \dots n\}$$

Comparing coefficients gives

$$(4.181)_{x_l^*} = (4.154)_{k=l} \quad P_{l+1}^{ix} = T_{l+1}^{ix} \Phi_l^x ; i \in \{2 \dots n\}$$

$$(4.181)_{\mu_l^m} = (4.155)_{k=l} \quad P_{l+1}^{im\mu} = T_{l+1}^{ix} \Phi_l^{m\mu} + T_{l+1}^{im\mu} ; i, m \in \{2 \dots n\}$$

$$(4.181)_{const.} = (4.156)_{k=l} \quad \alpha_{l+1}^i = T_{l+1}^{ix} \phi_l + t_{l+1}^i ; i \in \{2 \dots n\} \quad \square$$

Remark 17 To solve the Stackelberg game algorithmically, the following order of application of the equations of Theorem (10) is advisable ($i, j \in \{2, \dots, n\}$):

1. $M_T^{ix}, M_T^{ij\mu}, m_T^i$
2. $L_T^x, L_T^{i\mu}, l_T$
3. For k running backward from $T-1$ to 0
 - a) $N_k^{ix}, N_k^{ij\mu}$ and n_k^i
 - b) $T_{k+1}^{ix}, T_{k+1}^{ij\mu}, t_{k+1}^i$
 - c) $W_{k+1}^x, W_{k+1}^{i\mu}, w_{k+1}$
 - d) $\Phi_k^x, \Phi_k^{i\mu}, \phi_k$
 - e) $\Psi_k^{ix}, \Psi_k^{ij\mu}, \psi_k^i$
 - f) $M_k^{ix}, M_k^{ij\mu}, m_k^i$
 - g) $L_k^x, L_k^{i\mu}, l_k$

4. x_0^*, μ_0^i

5. For k running forward from 1 to T

a) $P_k^{1x}, P_k^{1i\mu}, \alpha_k^1$

b) $P_k^{ix}, P_k^{ij\mu}, \alpha_k^i$

c) u_k^{1*}, u_k^{i*}

d) x_k^*, μ_k^i

e) $g_k^1(x_k, u_k^1, \dots, u_k^n, x_{k-1}), g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1})$

6. $L^1(x_0, u^1, \dots, u^n), L^i(x_0, u^1, \dots, u^n)$

4.2.5 Special case: "One-induction" linear-quadratic games with one leader and one follower

In this subsection the results of the previous subsection (4.2.4) are specialized to a linear-quadratic 2-person game to allow comparison with Corollary (4) and Corollary (5) and to point out that the number and length of the equations of the game grow rapidly with the number of followers and the consideration of constant terms.

Corollary 6 *A 2-person linear-quadratic dynamic game (cf. Def. (4)) admits a unique open-loop Stackelberg equilibrium solution with one leader and one follower if*

- $Q_k^i \geq 0, R_k^{ii} > 0$ (defined for $k \in K, i \in N$).
- $(I - B_{k+1}^1 W_{k+1}^x - B_{k+1}^2 T_{k+1}^x)^{-1}$ and $(B_{k+1}^{2'}(Q_{k+1}^2 + L_{k+1}^\mu)B_{k+1}^2 + I - R_{k+1}^{12}B_{k+1}^{2'}M_{k+1}^\mu B_{k+1}^2)^{-1}$ (defined for $k \in K$) exist.

If these conditions are satisfied, the unique equilibrium strategies $\gamma_{k+1}^{i}(x_0)$ are given by (4.201), where the associated state trajectory x_{k+1}^* is given by (4.185).²⁵*

$$(4.182) \quad f_{k-1}(x_{k-1}, u_k^1, u_k^2) = A_k x_{k-1} + B_k^1 u_k^1 + B_k^2 u_k^2; k \in K$$

$$(4.183) \quad L^i(x_0, u^1, u^2) = \sum_{k=1}^T g_k^i(x_k, u_k^1, u_k^2, x_{k-1})$$

$$(4.184) \quad g_k^i(x_k, u_k^1, \dots, u_k^n, x_{k-1}) = \frac{1}{2}(x_k' Q_k^i x_k + u_k^{i'} u_k^i + u_k^{j'} R_k^{ij} u_k^j); k \in K; i, j \in \{1, 2\}; i \neq j$$

²⁵ For all equations belonging to this corollary and its proof, $k \in \{0, \dots, T-1\}$ if nothing different is stated.

$$(4.185) \quad x_{k+1}^* = \Phi_k^x x_k^* + \Phi_k^\mu \mu_k ; x_0^* = x_0$$

$$(4.186) \quad \Phi_k^x = (I - B_{k+1}^1 W_{k+1}^x - B_{k+1}^2 T_{k+1}^x)^{-1} A_{k+1}$$

$$(4.187) \quad \Phi_k^\mu = (I - B_{k+1}^1 W_{k+1}^x - B_{k+1}^2 T_{k+1}^x)^{-1} (B_{k+1}^1 W_{k+1}^\mu + B_{k+1}^2 T_{k+1}^\mu)$$

$$(4.188) \quad \mu_{k+1} = \Psi_k^x x_k^* + \Psi_k^\mu \mu_k ; \mu_0 = 0$$

$$(4.189) \quad \Psi_k^x = B_{k+1}^2 N_k^x \Phi_k^x$$

$$(4.190) \quad \Psi_k^\mu = A_{k+1} + B_{k+1}^2 (N_k^x \Phi_k^\mu + N_k^\mu)$$

$$(4.191) \quad W_{k+1}^x = -B_{k+1}^{1'} L_{k+1}^x - B_{k+1}^{1'} L_{k+1}^\mu N_k^x - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^x$$

$$(4.192) \quad W_{k+1}^\mu = -B_{k+1}^{1'} L_{k+1}^\mu (A_{k+1} + B_{k+1}^2 N_k^\mu) \\ - B_{k+1}^{1'} Q_{k+1}^2 A_{k+1} - B_{k+1}^{1'} Q_{k+1}^2 B_{k+1}^2 N_k^\mu$$

$$(4.193) \quad T_{k+1}^x = -B_{k+1}^{2'}(M_{k+1}^x + M_{k+1}^\mu B_{k+1}^2 N_k^x)$$

$$(4.194) \quad T_{k+1}^\mu = -B_{k+1}^{2'}M_{k+1}^\mu(A_{k+1} + B_{k+1}^2 N_k^\mu)$$

$$(4.195) \quad N_k^x = -(B_{k+1}^{2'}(Q_{k+1}^2 + L_{k+1}^\mu)B_{k+1}^2 + I - R_{k+1}^{12}B_{k+1}^{2'}M_{k+1}^\mu B_{k+1}^2)^{-1} \\ (B_{k+1}^{2'}L_{k+1}^x - R_{k+1}^{12}B_{k+1}^{2'}M_{k+1}^x)$$

$$(4.196) \quad N_k^\mu = -(B_{k+1}^{2'}(Q_{k+1}^2 + L_{k+1}^\mu)B_{k+1}^2 + I - R_{k+1}^{12}B_{k+1}^{2'}M_{k+1}^\mu B_{k+1}^2)^{-1} \\ (B_{k+1}^{2'}(Q_{k+1}^2 + L_{k+1}^\mu) - R_{k+1}^{12}B_{k+1}^{2'}M_{k+1}^\mu)A_{k+1}$$

$$(4.197) \quad M_k^x = Q_k^2 + A_{k+1}'[M_{k+1}^x \Phi_k^x + M_{k+1}^\mu \Psi_k^x]; M_T^x = Q_T^2$$

$$(4.198) \quad M_k^\mu = A_{k+1}'[M_{k+1}^x \Phi_k^\mu + M_{k+1}^\mu \Psi_k^\mu]; M_T^\mu = 0$$

$$(4.199) \quad L_k^x = Q_k^1 + A_{k+1}'L_{k+1}^x \Psi_k^x + A_{k+1}'L_{k+1}^\mu \Psi_k^x + A_{k+1}'Q_{k+1}^2 B_{k+1}^2 N_k^x \Psi_k^x; L_T^x = Q_T^1$$

$$(4.200) \quad L_k^\mu = A_{k+1}'L_{k+1}^x \Psi_k^\mu + A_{k+1}'L_{k+1}^\mu \Psi_k^\mu \\ + A_{k+1}'Q_{k+1}^2 A_{k+1} \mu_k + A_{k+1}'Q_{k+1}^2 B_{k+1}^2 (N_k^x \Psi_k^\mu + N_k^\mu); L_T^\mu = 0$$

$$(4.201) \quad \gamma_{k+1}^{i*}(x_0) = u_{k+1}^{i*} = P_{k+1}^{ix} x_k^* + P_{k+1}^{i\mu} \mu_k; i \in \{1, 2\}$$

$$(4.202) \quad P_{k+1}^{1x} = W_{k+1}^x \Phi_k^x$$

$$(4.203) \quad P_{k+1}^{1\mu} = W_{k+1}^x \Phi_k^\mu + W_{k+1}^\mu$$

$$(4.204) \quad P_{k+1}^{2x} = T_{k+1}^x \Phi_k^x$$

$$(4.205) \quad P_{k+1}^{2\mu} = T_{k+1}^x \Phi_k^\mu + T_{k+1}^\mu$$

PROOF:

Corollary (6) is proven in the same way as Theorem (10) taking into consideration simplifications resulting from the different number of followers and the modified state equation and cost functionals. \square

Remark 18 *Special attention should be paid to the fact that the assumption about the existence of unique solutions of the systems of equations (4.141), (4.142) and (4.143) in Theorem (10) is equivalent to assuming the existence of $(B_{k+1}^{2'}(Q_{k+1}^2 + L_{k+1}^\mu)B_{k+1}^2 + I - R_{k+1}^{12}B_{k+1}^{2'}M_{k+1}^\mu B_{k+1}^2)^{-1}$ in this special case.*

5 Conclusion

Beside some corrections of results for n -person discrete-time affine-quadratic dynamic games of prespecified fixed duration with open-loop and feedback information patterns in the literature, extensions were presented for the open-loop and feedback Stackelberg equilibrium solutions of n -person discrete-time affine-quadratic dynamic games of prespecified fixed duration concerning the number of followers and the possibility of algorithmic disintegration. These extensions enable a better modeling and numerical solution of real-world applications characterized by a hierarchical structure of the players' interactions.

Extending the number of leaders from 1 to n for open-loop and feedback Stackelberg discrete-time affine-quadratic dynamic games of prespecified fixed duration and finding an interrelation between the assumptions for the unique existence of Nash and Stackelberg equilibrium solutions for affine-quadratic dynamic games with open-loop and feedback information patterns and the matrices defining the affine-quadratic dynamic games are challenging tasks for future research.

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